On the character degrees and automorphism groups of finite p-groups by coclass

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der Technischen Universität Carolo-Wilhemina zu Braunschweig

zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigte Dissertation

von

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geboren am 17. Dezember 1985
in Wesel

Eingereicht am:	24. Juni 2013
Disputation am:	15. August 2013
1. Referentin:	Prof. Dr. Bettina Eick
2. Referent:	Prof. Dr. Gerhard Hiß

2013

Contents

Preface

Summary

In 1980 Leedham-Green & Newman [21] initialized the coclass theory, which has provided significant new insights into finite p-groups. For a finite p-group G of order p^n and nilpotency class $\mathrm{cl}(G)$ the coclass of G is defined as $\mathrm{cc}(G) := n - \mathrm{cl}(G)$. The classification of finite p-groups by their coclass is a challenging and fascinating open problem. A very important step has been made by Eick & Leedham-Green [9]. They introduced coclass families that are infinite families $(G_k \mid k \in \mathbb{N}_0)$ of finite p-groups of a fixed coclass with many interesting properties. They can be used to classify the 2-groups of fixed coclass and the 3-groups of coclass 1. By giving a counterexample we show that there is a mistake in [9]. We fill this gap here and thus complete the work in [9].
By definition all groups G_k of a coclass family $(G_k \mid k \in \mathbb{N}_0)$ can be described by a single parametrized presentation. We extend this to the automorphism groups $\mathrm{Aut}\, G_k$ and the number of irreducible characters of the groups G_k. For $l \in \mathbb{N}_0$ let $\mathrm{N}_l(G_k)$ denote the number of irreducible characters of G_k with degree p^l. One of the main results of this thesis is the following theorem.

Theorem. *Let $(G_k \mid k \in \mathbb{N}_0)$ be a coclass family and d denote the dimension of the associated infinite pro-p-group.*

- *There exists an integer b such that every irreducible character for every G_k has degree at most p^b.*
- *For every nonnegative integer $0 \leq l \leq b$ there exist a polynomial $f_l(x) \in \mathbb{Q}[x]$ with $deg(f_l) \leq d$ and a natural number w such that $\mathrm{N}_l(G_k) = f_l(p^k)$ for every $k \geq w$.*

Eick [7] considered the growth of the orders of the automorphism groups $\mathrm{Aut}\, G_k$. We continue her studies and we analyze the structure of the automorphism groups in detail. Let S denote the infinite pro-p-group associated to the coclass family $(G_k \mid k \in \mathbb{N}_0)$. We show that there exist a subgroup $A \leq \mathrm{Aut}\, S$, a normal series

$$A \triangleright B_0 \triangleright B_1 \triangleright B_2 \triangleright \cdots,$$

where B_{i+1} is the Frattini subgroup of B_i, an A-module N and a natural number c such that for $k \geq c$ the automorphism group $\mathrm{Aut}\, G_k$ is isomorphic to a group extension of N by A/B_{k-c}. By our definition of coclass families all groups G_k arise from one single parametrized presentation, in particular the group structures of all groups G_k can be decoded by the associated pro-p-group S and a single 2-cocycle. It turns out that the automorphism groups can be described in a similar way: there exists an element τ of the second cohomology group $H^2(A/B_0, N)$ which induces together with A a sequence $(\eta_k \mid k \geq c)$ of elements $\eta_k \in H^2(A/B_{k-c}, N)$ such that for $k \geq c$ the group extension defined by η_k is isomorphic to $\mathrm{Aut}\, G_k$.

Zusammenfassung

Leedham-Green und Newman [21] stellten 1980 die sogenannten Koklassenvermutungen auf, welche den Ausgangspunkt der Koklassentheorie bildeten. Mit Hilfe der Koklassentheorie konnten viele neue Einsichten in endliche p-Gruppen gewonnen werden. Die Koklasse einer endlichen p-Gruppe G der Ordnung p^n und der nilpotenten Länge $\mathrm{cl}(G)$ ist definiert als $\mathrm{cc}(G) := n - \mathrm{cl}(G)$. Die Klassifikation endlicher p-Gruppen anhand ihrer Koklasse ist ein forderndes und zugleich faszinierendes offenes Problem. Einen wichtigen Schritt in Richtung einer möglichen Klassifikation stellt die Veröffentlichung [9] von Eick & Leedham-Green dar. Sie führten Koklassenfamilien ein, welche unendliche Familien $(G_k \mid k \in \mathbb{N}_0)$ endlicher p-Gruppen gemeinsamer Koklasse sind. Mithilfe von Koklassenfamilien kann man die endlichen 3-Gruppen der Koklasse 1 und die 2-Gruppen der Koklasse r für festes $r \in \mathbb{N}$ klassifizieren. Die Publikation [9] von Eick & Leedham-Green enthält eine fehlerhafte Aussage. Wir zeigen dies, indem wir ein Gegenbeispiel angeben. Ferner beheben wir den Fehler und vervollständigen somit ihre Arbeit [9].
Per definitionem können alle Gruppen G_k einer Koklassenfamilie $(G_k \mid k \in \mathbb{N}_0)$ mit einer parametrisierten Gruppenpräsentation angegeben werden. In der vorliegenden Arbeit zeigen wir, dass analoge Aussagen für die Automorphismengruppen $\operatorname{Aut} G_k$ und die Anzahl der irreduziblen Charaktere der Gruppen G_k gelten. Für $l \in \mathbb{N}_0$ bezeichne $\mathrm{N}_l(G)$ die Anzahl der irreduziblen Charaktere von G_k vom Grad p^l. Eines der Hauptresultate dieser Arbeit ist folgender Satz:

Satz. *Es sei $(G_k \mid k \in \mathbb{N}_0)$ eine Koklassenfamilie und d bezeichne die Dimension der zugehörigen unendlichen pro-p-Gruppe.*

- *Es existiert eine ganze Zahl b, so dass für jedes k die irreduziblen Charaktere von G_k höchstens vom Grad p^b sind.*
- *Für jede nichtnegative ganze Zahl $0 \leq l \leq b$ existieren ein Polynom $f_l(x) \in \mathbb{Q}[x]$ vom Grad höchstens d und eine natürliche Zahl w so dass $\mathrm{N}_l(G_k) = f_l(p^k)$ für alle $k \geq w$ gilt.*

Eick [7] untersuchte das Wachstum der Gruppenordnungen der Automorphismengruppen. Wir führen ihre Studien fort und beschreiben zudem detailliert die Struktur der Automorphismengruppen $\operatorname{Aut} G_k$. Es sei S die zur Koklassenfamilie $(G_k \mid k \in \mathbb{N}_0)$ gehörende unendliche pro-p-Gruppe. Wir zeigen, dass eine Untergruppe $A \leq \operatorname{Aut} S$, eine Normalreihe

$$A \rhd B_0 \rhd B_1 \rhd B_2 \rhd \cdots ,$$

wobei B_{i+1} die Frattiniuntergruppe von B_i ist, ein A-Modul N und eine natürliche Zahl c existieren, so dass für $k \geq c$ die Automorphismengruppe $\operatorname{Aut} G_k$ isomorph zu einer Gruppenerweiterung von N mit A/B_{k-c} ist. Alle Gruppen G_k können mit einer einzelnen parametrisierten Gruppenpräsentation dargestellt werden. Diese Präsentation wird eindeutig bestimmt durch die zur Koklassenfamilie gehörende unendliche pro-p-Gruppe und einen 2-Kozykel. Wir geben eine analoge Beschreibung der Automorphismengruppen $\operatorname{Aut} G_k$ an: Es existiert ein Element τ der zweiten Kohomologiegruppe $H^2(A/B_0, N)$, welches mit A eine Folge $(\eta_k \mid k \geq c)$ von Elementen η_k aus $H^2(A/B_{k-c}, N)$ induziert, so dass für $k \geq c$ die durch η_k definierte Gruppenerweiterung isomorph zu $\operatorname{Aut} G_k$ ist.

Danksagung

Zunächst möchte ich mich bei Prof. Dr. Bettina Eick dafür bedanken, dass ich die Möglichkeit der Promotion erhalten habe. Von ihr kam auch der Vorschlag sich mit den Automorphismengruppen von Koklassenfamilien zu befassen. Desweiteren half sie mir bei der Vereinfachung von Formulierungen und der Notation. Ich schätzte auch das offene Arbeitsklima im Institut.
Ich bedanke mich bei Prof. Dr. Gerhard Hiß, welcher den Kontakt zwischen Prof. Dr. Bettina Eick und mir herstellte. Zudem bin ich sehr dankbar für die Ausbildung, die mir in Aachen zu teil wurde.
Zum Dank verpflichtet bin ich desweiteren Prof. Dr. Max Horn. Die Gespräche mit ihm haben mich vor manch einer Sackgasse bewahrt. Sein offenes Ohr und sein Zuspruch haben mir viel bedeutet.
Teile der Arbeit wurden von verschiedenen Personen Korrektur gelesen. Hierfür danke ich: Dr. Andreas Distler, Prof. Dr. Bettina Eick, Dr. Dörte Feichtenschlager und Prof. Dr. Max Horn.
Darüber hinaus möchte ich mich bei meiner Familie dafür bedanken, dass sie mich bei meinen eingeschlagenen Weg unterstützt hat.
Den größten Dank gebührt sicherlich meiner Partnerin Ina. Sie hat in den letzten drei Jahren viel Verständnis gezeigt und mich in vielerlei Hinsicht unterstützt.

Chapter 1

Introduction

Since the early days of group theory mathematicians have classified finite groups by their order. For some orders, this can easily be done, for instance for a prime p there exists only one isomorphism class of groups of order p. In general, the difficulty of determining all groups of a given finite order $n \in \mathbb{N}$ up to isomorphism is increasing with the number of factors in the prime factorization of n. In 2002, Besche, Eick & O'Brien [2] accomplished the construction of all finite groups of order at most 2000 up to isomorphism. The constructed groups are available in the computer algebra systems GAP [11] and MAGMA [3]. It is remarkable that over 99% of the more than 49 billion constructed groups have order 2^{10}. Hence it may not be surprising that the classification of groups of prime power order is a difficult problem.
Leedham-Green & Newman introduced the coclass of a prime power group. For a prime p and a group G of order p^n and nilpotency class $\mathrm{cl}(G)$ the coclass of G is defined as $\mathrm{cc}(G) := n - \mathrm{cl}(G)$. It seems more promising to consider finite groups of prime power order by their coclass instead of their order.

Coclass theory

We write p to denote a prime and we call finite groups of p-power order finite p-groups. Coclass theory has its origins in the study of p-groups of maximal nilpotency class by Blackburn in 1958: a group of order p^n has maximal class if $n - 1$ is the nilpotency class. In 1980 Leedham-Green & Newman [21] extended this studies by introducing the coclass. Further, they published the so-called coclass conjectures which formed for a long time the backbone of coclass theory. The conjectures, which are stated below, are closely related to the structure of pro-p-groups of finite coclass. A pro-p-group of finite coclass $r \in \mathbb{N}$ is the inverse limit of finite p-groups of coclass r [20, Chapter 7].

Coclass Conjecture A. For some function $f(p, r)$, every finite p-group of coclass r has a normal subgroup K of nilpotency class at most 2 and index at most $f(p, r)$. If $p = 2$, one can require K to be abelian.

Coclass Conjecture B. For some function $g(p, r)$, every finite p-group of coclass r has derived length at most $g(p, r)$.

Coclass Conjecture C. Every pro-p-group of finite coclass is solvable.

Coclass Conjecture D. For fixed prime p and natural number r there are only finitely many isomorphism classes of infinite pro-p-groups of coclass r.

Coclass Conjecture E. There are only finitely many isomorphism classes of infinite solvable pro-p-groups of coclass r.

The coclass conjectures are ordered by their strength, where Coclass Conjecture A is the most general result yielding the other four. The work of several mathematicians led to a proof of Coclass Conjecture A in 1994 [19, 29].

The well-known structure of 2-groups of coclass 1 has been a source of inspiration for many conjectures in coclass theory. Up to isomorphism there exists only one infinite pro-2-group D_∞ of coclass 1. We write D_{2^k} to denote the dihedral group of order 2^k and we use a similar notation for the quaternion groups Q_{2^k} and the semi-dihedral groups SD_{2^k}. Almost all finite 2-groups of coclass 1 lie in one of the families $\mathcal{F}_1 := (D_{2^{k+3}} \mid k \in \mathbb{N}_0)$, $\mathcal{F}_2 := (Q_{2^{k+4}} \mid k \in \mathbb{N}_0)$ and $\mathcal{F}_3 := (SD_{2^{k+5}} \mid k \in \mathbb{N}_0)$ up to isomorphism. These families derive from the pro-2-group D_∞ as outlined in the following paragraph. For a group G we write $\gamma_i(G)$ to denote the i-th term of the lower central series of G, that is, $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. Then $D_\infty/\gamma_1(D_\infty)$ is isomorphic to the Klein four-group V_4 and $D_\infty/\gamma_k(D_\infty) \cong D_{2^{k+1}}$ for $k > 1$. The groups Q_{2^k} and SD_{2^k} are group extensions of the cyclic group C_2 of order 2 by $D_{2^{k-1}}$. Additionally, for each family $\mathcal{F}_i = (G_k \mid k \in \mathbb{N}_0)$ the groups G_k can be described by a single parametrized presentation, for instance

$$D_{2^{k+3}} \cong \langle a, b \mid a^2 = 1, b^{2^{k+2}} = 1, b^a = b^{-1} \rangle.$$

The families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are examples for so-called coclass families. These were introduced by Eick & Leedham-Green [9] and are infinite families of finite p-groups induced by certain parametrized presentations. For each prime p and natural number r there exist coclass families consisting of finite p-groups of coclass r. Coclass families have many interesting properties, for instance they can be used to classify the 2-groups of fixed coclass and the 3-groups of coclass 1.

Theorem 1 ([9, Remark 4])**.** *There exist finitely many coclass families that contain almost all p-groups of coclass $r \in \mathbb{N}$ up to isomorphism, if and only if $p = 2$ or $(p, r) = (3, 1)$.*

Every coclass family $\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$ is associated to an infinite pro-p-group S of finite coclass. To be more precise, let $\mathbb{Z}_p$ denote the p-adic integers and let

$$\gamma_1(S) \geq \gamma_2(S) \geq \gamma_3(S) \geq \cdots$$

be the lower central series of S. Then there exist natural numbers $j > i$ such that $\gamma_i(S)$ is a free $\mathbb{Z}_p$-module of finite rank d, say, and G_k is a group extension of $\gamma_i(S)/\gamma_j(S)^{p^k}$ by $S/\gamma_i(S)$ for every $k \geq 0$. The natural number d is also called the dimension of S. For an explicit definition of coclass families we refer to Chapter 5.

We say that a coclass family $\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$ is isomorphic to another one $\mathcal{F}' = (H_k \mid k \in \mathbb{N}_0)$ if there exists an integer c such that $G_k \cong H_{k+c}$ for all but finitely many $k \in \mathbb{N}_0$ with $k \geq -c$. Due to a result of Eick & Leedham-Green [9] the isomorphism class of a coclass family $\mathcal{F}$ is determined by finitely many groups of $\mathcal{F}$. By giving a counterexample we show that there is a mistake in the corresponding proof in [9]. We fill this gap here and thus complete the work in [9].

Character degrees and automorphism groups of coclass families

The main focus of this thesis lies on the character degrees and automorphism groups of groups G_k of a coclass family $\mathcal{F} = (G_k \mid k \in \mathbb{N}_0)$. By definition all groups G_k of $\mathcal{F}$ can be defined by a single parametrized presentation. The question arises if this can be extended to invariants of the groups G_k. Eick & Feichtenschlager [8] have considered this for the Schur multipliers $M(G_k)$.

Theorem 2 ([8, Theorem 2])**.** *Let $(G_k \mid k \in \mathbb{N}_0)$ be a coclass family. Then there exist m, r_1, …, $r_m \in \mathbb{N}_0$ and s_1, …, $s_m \in \mathbb{Z}$ so that for almost all $k \in \mathbb{N}_0$ the Schur multiplier $M(G_k)$ has the form*

$$M(G_k) \cong C_{p^{r_1 k + s_1}} \times \cdots \times C_{p^{r_m k + s_m}}.$$

It has also been conjectured that the groups G_k have isomorphic mod-p cohomology rings. In the following sections we outline our main results: we show that the character degrees of almost all groups G_k can be described by finitely many polynomials and that the automorphism groups $\operatorname{Aut} G_k$ are derived from an infinite group and a 2-cocycle.

Character degrees

It is well-known that all irreducible characters of a finite p-group are monomial; that is, each character is induced from a linear character of some subgroup. However, despite this significant structural result, very little is known about the character tables of finite p-groups. Isaacs and Passmann [14, 15] initiated a study of groups with certain character degrees. Berkovich [1, 3.50] and Mann [22] provided further results. We refer to the survey article by Mann [23] for references and a brief survey on the character theory of finite p-groups.
In this thesis we consider the character theory of finite p-groups using the coclass as main invariant. For $l \in \mathbb{N}_0$ and a group G let $\mathrm{N}_l(G)$ denote the number of irreducible characters of G with degree p^l. One of our main results is the following theorem. It is proven in Chapter 8.

Theorem 3. *Let $(G_k \mid k \in \mathbb{N}_0)$ be a coclass family and d denote the dimension of the associated pro-p-group.*

- *There exists an integer b such that every irreducible character for every G_k has degree at most p^b.*
- *For every nonnegative integer $0 \leq l \leq b$ there exist a polynomial $f_l(x) \in \mathbb{Q}[x]$ with $\deg(f_l) \leq d$ and a natural number w such that $\mathrm{N}_l(G_k) = f_l(p^k)$ for every $k \geq w$.*

Hence Theorem 1 implies that the character degrees for the 2-groups of coclass r for fixed r and for all 3-groups of coclass 1 can be described in finite terms.
In order to prove Theorem 3 we first consider finite p-groups in general. Mackey's Theorem yields group theoretical conditions for the irreducibility and equality of characters induced from linear characters of subgroups. This is outlined in Chapter 7 and leads to a description of $\mathrm{N}_l(G)$. We then apply this to coclass families in Chapter 8.

Automorphism groups

The famous divisibility conjecture states that if G is a finite, noncyclic p-group with $|G| > p^2$ then $|G|$ divides the order of the automorphism group $\mathrm{Aut}(G)$. Eick [7] has addressed the divisibility conjecture for 2-groups by studying the growth of the orders of the automorphism groups $\operatorname{Aut} G_k$ of a coclass family $(G_k \mid k \in \mathbb{N}_0)$.

Theorem 4 ([7, Theorem 2]). *Let $(G_k \mid k \in \mathbb{N}_0)$ be a coclass family of 2-groups. Then there exist natural numbers l and f such that $|\operatorname{Aut} G_k|/|G_k| = |\operatorname{Aut} G_{k+1}|/|G_{k+1}| \cdot 2^l$ for every $k \geq f$.*

By Theorem 1, for every $r \in \mathbb{N}$ there exist finitely many coclass families that contain almost all 2-groups of coclass r up to isomorphism. Hence Theorem 4 yields

Theorem 5 ([7]). *For every $s \in \mathbb{N}$ there exists $o(r,s) \in \mathbb{N}$ such that $2^s|G|$ divides $|\operatorname{Aut} G|$ for all 2-groups G of coclass r and order at least $o(r,s)$.*

Although Theorem 4 can easily be generalized to all primes, it is unknown if Theorem 5 holds for odd primes.
In Chapter 9 we continue the work of Eick [7] and we analyze the structure of the automorphism groups in detail. Let $\mathcal{F} = (G_k \mid k \geq 0)$ be a coclass family with associated pro-p-group S. We show that there exist a subgroup $A \leq \operatorname{Aut} S$, a normal series

$$A \rhd B_0 \rhd B_1 \rhd B_2 \cdots ,$$

where B_{i+1} is the Frattini subgroup of B_i, an A-module N and a natural number c such that for $k \geq c$ the automorphism group $\operatorname{Aut} G_k$ is isomorphic to a group extension of N by A/B_{k-c}. By our definition of coclass families all groups G_k arise from one single parametrized presentation, in particular the group structures of all groups G_k can be decoded by the associated pro-p-group S and a single 2-cocycle. It turns out that the automorphism groups can be described in a similar way: there exists an element τ of the second cohomology group $H^2(A/B_0, N)$ which induces together with A a sequence $(\eta_k \mid k \geq c)$ of elements $\eta_k \in H^2(A/B_{k-c}, N)$ such that for large k the group extension defined by η_k is isomorphic to $\operatorname{Aut} G_k$.

Outline of this thesis

This thesis is divided into three parts.

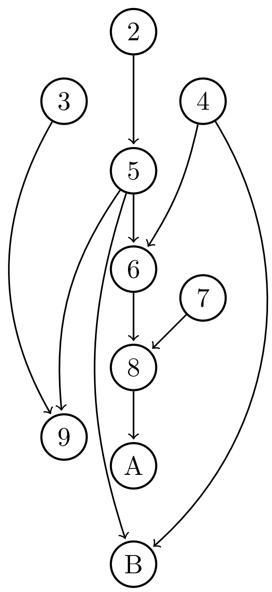
In Part I we gather and develop the necessary tools to handle coclass families.

The basic notations and definitions of cohomology groups are given in Chapter 2. Further, we recall some results of Eick & Leedham-Green [9] dealing with cohomology groups of torsion-free modules. We give alternative proofs enabling us to improve bounds.

Following [25] we consider automorphism groups of group extensions in Chapter 3.

In computer algebra systems such as GAP finite p-groups are implemented as polycyclic groups. We briefly summarize some basic definitions and results for polycyclic groups with a focus on group extensions in Chapter 4.

Eick & Leedham-Green [9] have shown that periodic patterns in shaved coclass trees arise from coclass families. By giving a counterexample in Appendix B we show that their article contains an incorrect Theorem [9, Theorem 23]. Further, we make sure that their main results in [9] are still valid. For this purpose we analyze the compatible pairs considered in [9, Theorem 23]. This is done in Subsection 5.3.1 and leads to sharper bounds.



Part I: Preliminaries
- 2: Cohomology groups
- 3: Automorphism groups of group extensions
- 4: Polycyclic groups
- 5: Coclass Theory
- 6: Parametrized families
- 7: Irreducible characters of finite p-groups

Part II:
- 8: Character degrees of a coclass family

Part III:
- 9: Automorphism groups of finite p-groups by coclass

Appendix
- A: Character degrees of 2-groups of coclass 2
- B: Counterexample to Theorem 23 of [9]

Figure 1.1: Dependances of the chapters in this thesis

In Chapter 5 parametrized families are introduced, which naturally appear in the study of character degrees of groups in a coclass family. We show that the class of parametrized families is closed under some group constructions such as taking intersections.

Then in Chapter 7 we introduce a setup to address character degrees of finite p-groups.

The character degrees of groups in a coclass family are considered in Part II, which includes a proof of Theorem 3.

Structure theorems for automorphism groups of groups in coclass families are developed in Part III.

In Figure 1.1 the dependencies of the chapters are visualized.

Notation

In this thesis all groups act from the right. For a group G and a G-set A we denote the image of $a \in A$ under $g \in G$ by a^g. If additionally A is abelian, we write $a.g$ instead of a^g.
For two functions α and β the composition of β and α is denoted by $\alpha \circ \beta$, that is, $\alpha \circ \beta(x) = \alpha(\beta(x))$.

Part I

Preliminaries

Chapter 2

Cohomology groups

Group cohomology is one essential tool to deal with group extensions. In Section 2.1 we give some basic definitions and notations, and we mention the well-known group theoretical interpretation of the second cohomology group. Then in Section 2.2 a long exact sequence of cohomology groups is recalled. Finally, we generalize a result by Eick & Leedham-Green [7] in Subsection 2.2.1.
Throughout this chapter, let G be a group and A a G-module.

2.1 Cochain maps

In this section we give a definition of cohomology groups following [26, p. 338f.]. For $n \in \mathbb{N}$ we write G^n to denote the outer direct product $G \times \cdots \times G$ with n factors. A map $\gamma : G^n \to A$ is called *n-cochain map*, if $\gamma(x) = 0$ for every $x = (x_1, \ldots, x_n) \in G^n$ with $x_i = 1$ for some $1 \leq i \leq n$. The set of n-cochain maps endowed with the group addition $\gamma + \gamma' : G^n \to A$, $x \mapsto \gamma(x) + \gamma'(x)$ is an abelian group and denoted by $C^n(G, A)$. The *coboundary homomorphism* d^n is defined as the group homomorphism

$$d^n : C^n(G, A) \to C^{n+1}(G, A), \gamma \mapsto d^n(\gamma),$$

where $d^n(\gamma) \in C^{n+1}(G, A)$ maps $(g_1, \ldots, g_{n+1})$ to

$$\gamma(g_2, \ldots, g_{n+1}) + \sum_{j=1}^{n} (-1)^j \cdot \gamma(g_1, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \ldots, g_{n+1}) + (-1)^{n+1} \cdot \gamma(g_1, \ldots, g_n).g_{n+1}.$$

For $n = 0$ put $C^0(G, A) := A$ and define $d^0 : C^0(G, A) \to C^1(G, A)$ by setting $d^0(a) : G \to A$, $g \mapsto a.g - a$ for $a \in A = C^0(G, A)$. Then

$$C^0(G, A) \xrightarrow{d_0} C^1(G, A) \xrightarrow{d_1} C^2(G, A) \xrightarrow{d^2} \cdots$$

is a cochain complex, that is, the composition $d^{n+1} \circ d^n$ is trivial, in other words $\operatorname{Im} d^n \leq \operatorname{Ker} d^{n+1}$. For $n \in \mathbb{N}_0$ define $Z^n(G, A) := \operatorname{Ker} d^n$, $B^0(G, A) := 0$ and $B^n(G, A) := \operatorname{Im} d^{n-1}$ if $n \geq 1$. The elements of $Z^n(G, A)$ and $B^n(G, A)$ are called *n-cocycles* and *n-coboundaries*, respectively. Then the n-th *cohomology group* $H^n(G, A)$ is defined as the factor group $Z^n(G, A)/B^n(G, A)$.

Let us consider an application of group cohomology in group theory. An extension $0 \to A \stackrel{\iota_1}{\to} E_1 \stackrel{\vartheta_1}{\to} G \to 1$ of the G-module A by G is said to be equivalent to another extension $0 \to A \stackrel{\iota_2}{\to} E_2 \stackrel{\vartheta_2}{\to} G \to 1$, if there exists a group isomorphism $\theta : E_1 \to E_2$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota_1"] \arrow[d, equal, "\mathrm{id}"] & E_1 \arrow[r, "\vartheta_1"] \arrow[d, "\theta"] & G \arrow[r] \arrow[d, equal, "\mathrm{id}"] & 1 \\
0 \arrow[r] & A \arrow[r, "\iota_2"] & E_2 \arrow[r, "\vartheta_2"] & G \arrow[r] & 1
\end{tikzcd}$$

is commutative. The corresponding equivalence classes can be described by the second cohomology group. For a 2-cocycle $\gamma \in Z^2(G, A)$ let $E(\gamma)$ denote the group extension of A by G via γ; that is, $E(\gamma)$ is the product set $G \times A$ endowed with the group multiplication $(g,t)(h,s) = (gh, t.h + s + \gamma(g,h))$. Further, let ι_γ and ϑ_γ be the embedding $\iota_\gamma : A \to E(\gamma)$, $a \mapsto (1, a)$ and the natural homomorphism $\vartheta_\gamma : E(\gamma) \to G$, $(g, a) \mapsto g$. Then γ induces the extension $0 \to A \stackrel{\iota_\gamma}{\to} E(\gamma) \stackrel{\vartheta_\gamma}{\to} G \to 1$.

Theorem 6 ([4, (3.12) Theorem]). *Let $\mathcal{E}$ be the set of equivalence classes of extensions $0 \to A \stackrel{\iota}{\to} E \stackrel{\vartheta}{\to} G \to 1$. Then the map from $H^2(G, A)$ to $\mathcal{E}$ which maps $\gamma + B^2(G, A)$ to the equivalence class of the extension induced by γ is a bijection.*

2.2 Exact sequences of cohomology groups

In the first part of this section we consider a well-known long exact sequence of cohomology groups, see for instance [20, p. 196f.]. We choose a very explicit definition of this exact sequence. Then in Subsection 2.2.1 we generalize a result of Eick & Leedham-Green [7, Theorem 18] concerning cohomology groups of quotients of torsion-free modules.
As an elementary tool we need

Lemma 7 (Snake Lemma [18, Lemma 9.1]). *Let*

$$\begin{tikzcd}
 & X_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & X_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & Y_1 \arrow[r, "h_1"] & Y_2 \arrow[r, "h_2"] & Y_3 &
\end{tikzcd} \tag{2.2.1}$$

be a commutative diagram of abelian groups and group homomorphisms with exact rows.

(i) *The sequences* $\operatorname{Ker} f_1 \xrightarrow{a_1} \operatorname{Ker} f_2 \xrightarrow{a_2} \operatorname{Ker} f_3$ *and* $\operatorname{Coker} f_1 \xrightarrow{b_1} \operatorname{Coker} f_2 \xrightarrow{b_2} \operatorname{Coker} f_3$*, where a_1, a_2 and b_1, b_2 are induced by g_1, g_2 and h_1, h_2, respectively, are exact.*

(ii) *Let δ be the map* $\operatorname{Ker} f_3 \to \operatorname{Coker} f_1$ *constructed as follows: for $a \in \operatorname{Ker} f_3$ let $\tilde{a} \in X_2$ be such that $g_2(\tilde{a}) = a$ and put $\delta(a) := h_1^{-1} \circ f_2(\tilde{a}) + \operatorname{Im} f_1$. Then the map δ is a well-defined group homomorphism making the following sequence exact.*

$$\operatorname{Ker} f_1 \xrightarrow{a_1} \operatorname{Ker} f_2 \xrightarrow{a_2} \operatorname{Ker} f_3 \xrightarrow{\delta} \operatorname{Coker} f_1 \xrightarrow{b_1} \operatorname{Coker} f_2 \xrightarrow{b_2} \operatorname{Coker} f_3$$

The homomorphism δ is called connecting homomorphism.

The proof of the Snake Lemma is technical but straightforward. Hence in [18] it is left to the reader as an exercise. For the sake of completeness we give a proof.

Proof. ad (i): Since Diagram 2.2.1 commutes, it is straightforward to observe that $g_1(\operatorname{Ker} f_1) \subseteq \operatorname{Ker} f_2$ and $g_2(\operatorname{Ker} f_2) \subseteq \operatorname{Ker} f_3$. Hence the homomorphisms $a_1 : \operatorname{Ker} f_1 \to \operatorname{Ker} f_2$, $x \mapsto g_1(x)$ and $a_2 : \operatorname{Ker} f_2 \to \operatorname{Ker} f_3$, $x \mapsto g_2(x)$ are well-defined. Now let x be an element of $\operatorname{Ker} a_2$. By the exactness of the rows in Diagram 2.2.1 the kernel of a_2 equals $\operatorname{Ker} g_2 \cap \operatorname{Ker} f_2 = \operatorname{Im} g_1 \cap \operatorname{Ker} f_2$. Hence there exists $y \in X_1$ with $g_1(y) = x$ and by the commutativity of Diagram 2.2.1 we have $0 = f_2 \circ g_1(y) = h_1 \circ f_1(y)$. Since h_1 is injective, the element y lies in $\operatorname{Ker} f_1$ and hence $x = g_1(y) = a_1(y)$. It follows that $\operatorname{Im} a_1 = \operatorname{Ker} a_2$, in other words the sequence $\operatorname{Ker} f_1 \xrightarrow{a_1} \operatorname{Ker} f_2 \xrightarrow{a_2} \operatorname{Ker} f_3$ is exact. We move on to the maps b_1 and b_2. The commutativity of Diagram 2.2.1 yields $h_1(\operatorname{Im} f_1) \leq \operatorname{Im} f_2$ and $h_2(\operatorname{Im} f_2) \leq \operatorname{Im} f_3$ and as a result the homomorphisms $b_1 : \operatorname{Coker} f_1 \to \operatorname{Coker} f_2$, $x + \operatorname{Im} f_1 \mapsto h_1(x) + \operatorname{Im} f_2$ and $b_2 : \operatorname{Coker} f_2 \to \operatorname{Coker} f_3$, $x + \operatorname{Im} f_2 \mapsto h_2(x) + \operatorname{Im} f_3$ are well-defined. Now let $x + \operatorname{Im} f_2 \in \operatorname{Ker} b_2$, that is, $h_2(x) \in \operatorname{Im} f_3$. The surjectivity of g_2 yields the existence of an element $y \in X_2$ with $f_3 \circ g_2(y) = h_2(x)$. Since $f_3 \circ g_2(y) = h_2 \circ f_2(y)$, we have $x - f_2(y) \in \operatorname{Ker} h_2 = \operatorname{Im} h_1$. It follows that $\operatorname{Im} b_1 = \operatorname{Ker} b_2$, in other words, the sequence $\operatorname{Coker} f_1 \xrightarrow{b_1} \operatorname{Coker} f_2 \xrightarrow{b_2} \operatorname{Coker} f_3$ is exact.
ad (ii): By the commutativity of 2.2.1 we have $0 = f_3(a) = f_3 \circ g_2(\tilde{a}) = h_2 \circ f_2(\tilde{a})$, in other words $f_2(\tilde{a}) \in \operatorname{Ker} h_2 = \operatorname{Im} h_1$. Further, we have $\operatorname{Ker} g_2 = \operatorname{Im} g_1$. Then $f_2 \circ g_1 = h_2 \circ f_1$ yields $h_1^{-1} \circ f_2(\operatorname{Ker} g_2) = \operatorname{Im} f_1$. Thus δ is well-defined. It remains to show that $\operatorname{Im} \delta = \operatorname{Ker} b_1$. Let $g + \operatorname{Im} f_1 \in \operatorname{Ker} b_1$, this is equivalent to $h_1(g) \in \operatorname{Im} f_2$. Hence there exists $\tilde{a} \in X_2$ with $f_2(\tilde{a}) = h_1(g)$, in particular $g = h_1^{-1} \circ f_2(\tilde{a})$. Put $a := g_2(\tilde{a})$. Since $f_3 \circ g_2(\tilde{a}) = h_2 \circ f_2(\tilde{a}) = h_2 \circ h_1(g) = 0$, the element a lies in the kernel of f_3. It follows that $\operatorname{Im} \delta = \operatorname{Ker} b_1$.

□

In what follows, we apply the Snake Lemma to cohomology groups. Let

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\vartheta} C \to 0$$

be a short exact sequence of G-modules. In particular ι and ϑ are a G-module monomorphism and a G-module epimorphism, respectively. Then for $n \geq 0$ the short exact sequence induces the following group homomorphisms

$$\begin{array}{ccc} C^n(G,A) & \to & C^n(G,B) \\ \gamma & \mapsto & \iota \circ \gamma \end{array} \quad \text{and} \quad \begin{array}{ccc} C^n(G,B) & \to & C^n(G,C) \\ \gamma & \mapsto & \vartheta \circ \gamma \end{array}.$$

Evidently, the maps ι and ϑ also induce group homomorphisms of n-cocycles. This leads us to the following commutative diagram

$$\begin{tikzcd} & C^n(G,A) \arrow[r] \arrow[d, "d_A^n"] & C^n(G,B) \arrow[r] \arrow[d, "d_B^n"] & C^n(G,C) \arrow[r] \arrow[d, "d_C^n"] & 0 \\ 0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] & Z^{n+1}(G,B) \arrow[r] & Z^{n+1}(G,C) & \end{tikzcd}$$

where the rows are induced by the short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\vartheta} C \to 0$ and the vertical arrows are the coboundary maps. Note that the rows of the diagram are exact and that for $D \in \{A, B, C\}$ the kernel and the cokernel of d_D^n are $Z^n(G,D)$ and $H^{n+1}(G,D)$, respectively. Let $\delta_n : Z^n(G,C) \to H^{n+1}(G,A)$ be the map defined by

$$\delta_n(\gamma) := \iota^{-1} \circ d^n(\tilde{\gamma}) + B^{n+1}(G,A),$$

where $\tilde{\gamma} \in C^{n+1}(G,B)$ denotes an element with $\vartheta \circ \tilde{\gamma} = \gamma$. Then by the Snake Lemma the sequences $Z^n(G,A) \to Z^n(G,B) \to Z^n(G,C)$ and $H^{n+1}(G,A) \to H^{n+1}(G,B) \to H^{n+1}(G,C)$ induced by ι and ϑ are exact. Further, the map δ_n is a well-defined group homomorphism making the sequence

$$Z^n(G,A) \longrightarrow Z^n(G,B) \longrightarrow Z^n(G,C) \xrightarrow{\delta_n} H^{n+1}(G,A) \longrightarrow H^{n+1}(G,B) \longrightarrow H^{n+1}(G,C) \quad (2.2.2)$$

exact. This leads for $n \geq 0$ to the exact sequence

$$H^n(G,A) \longrightarrow H^n(G,B) \longrightarrow H^n(G,C) \longrightarrow H^{n+1}(G,A) \longrightarrow H^{n+1}(G,B) \longrightarrow H^{n+1}(G,C) \quad (2.2.3)$$

which can be combined to the well-known long exact sequence

$$0 \to H^0(G,A) \to \cdots \to H^n(G,A) \to H^n(G,B) \to H^n(G,C) \to H^{n+1}(G,A) \to \cdots \quad (2.2.4)$$

2.2.1 Cohomology groups of quotients of torsion-free modules

In this subsection, we assume that the group G is finite and that the short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\vartheta} C \to 0$ of G-modules consists of an embedding $\iota : A \hookrightarrow B$ and a natural homomorphism $\vartheta : B \twoheadrightarrow C$, whence A is a G-submodule of B and C is the factor module B/A. For a group U write $\exp U$ to denote the exponent of U. Fix $n \in \mathbb{N}_0$ and denote

$$e := \exp H^n(G,B) \text{ and } f := \exp H^{n+1}(G,A).$$

The following lemma generalizes [9, Theorem 16] and its proof follows the original proof.

Lemma 8. *Assume that B is torsion-free and A is a submodule of $f \cdot B$. Then the Exact Sequence (2.2.2) gives rise to an exact sequence*

$$Z^n(G,B) \longrightarrow Z^n(G,B/A) \longrightarrow H^{n+1}(G,A) \to 0. \quad (2.2.5)$$

If additionally $A \leq e \cdot B$, then the Exact Sequence (2.2.5) leads to a short exact sequence

$$0 \to H^n(G,B) \longrightarrow H^n(G,B/A) \longrightarrow H^{n+1}(G,A) \to 0. \quad (2.2.6)$$

Proof. For $l \in \mathbb{N}_0$ let ι_l denote the group homomorphism $\iota_l : H^l(G,A) \to H^l(G,B)$ induced by $A \hookrightarrow B$. Put $D := \{b \in B \mid f \cdot b \in A\}$ and note that D is a G-submodule of $A \leq f \cdot B$, and $f \cdot D$ equals A. Thus ι_{n+1} decomposes into $H^{n+1}(G, f \cdot D) \to H^{n+1}(G,D) \to H^{n+1}(G,B)$, where each homomorphism is induced by an embedding. Since D is torsion-free, the G-module D is isomorphic to $f \cdot D = A$ and the exponent of $H^{n+1}(G,D)$ equals $f = \exp H^{n+1}(G,A)$. It follows that the group homomorphism $H^{n+1}(G, f \cdot D) \to H^{n+1}(G,D)$ is trivial. Hence ι_{n+1} is also trivial and the Exact Sequence (2.2.2) induces the Exact Sequence (2.2.5).
Now assume that A is also a subgroup of $e \cdot B$. Then ι_n is the composition $H^n(G,A) \to H^n(G,eB) \to H^n(G,B)$. As $H^n(G,eB) \to H^n(G,B)$ is trivial, the result follows from (2.2.3). □

Recall that $\mathbb{Z}_p$ denotes the p-adic integers. The following Lemma 9 is a generalized result of Eick & Leedham-Green [9, Theorem 18]. For the proof of Lemma 9 we have chosen another approach enabling us to give sharper bounds.

Lemma 9. *Assume that B is torsion-free, $A \leq f \cdot B$ and A is a free $\mathbb{Z}_p$-module of finite rank. Let ϑ_n denote the group homomorphism $Z^n(G,B) \to Z^n(G,B/A)$ in the Exact Sequence (2.2.5).*

(i) *The Exact Sequence (2.2.5) gives rise to a split short exact sequence*

$$0 \to \operatorname{Im}\vartheta_n \longrightarrow Z^n(G,B/A) \longrightarrow H^{n+1}(G,A) \to 0.$$

(ii) *If additionally $A \leq e \cdot B$, then the Short Exact Sequence (2.2.6) splits.*

Proof. ad (i): By assumption there exists a $\mathbb{Z}_pG$-submodule $D \leq B$ such that $f \cdot D = A$. Since G is finite and $A = f \cdot D \cong D$ is finitely generated, the $\mathbb{Z}_p$-modules $C^n(G,D)$ and $C^{n+1}(G,D)$ are also finitely generated. It follows that $H^{n+1}(G,A) \cong H^{n+1}(G,D)$ is finite. Further, by Lemma 8 the factor group $Z^n(G,B/A)/\operatorname{Im}\vartheta_n$ is isomorphic to $H^{n+1}(G,A)$. Hence it remains to show the existence of a submodule $K \leq Z^n(G,B/A)$ with $K \cap \operatorname{Im}\vartheta_n = 0$ and $K \cong H^{n+1}(G,A)$. Recall that $Z^n(G,D)$ is the kernel of the coboundary map $d^n : C^n(G,D) \to C^{n+1}(G,D)$, which is a homomorphism of finitely generated $\mathbb{Z}_p$-modules. Let ν_p denote the p-adic valuation. By the existence of the Smith normal form of the coboundary map there are a $\mathbb{Z}_p$-module basis of $C^n(G,D)$, say $a_1,\ldots,a_m$, and corresponding p-adic integers, say $b_1,\ldots,b_m$, such that the module $Z^n(G,D/f\cdot D) = \{\delta \in C^n(G,D) \mid d^n\delta \in C^{n+1}(G,f\cdot D)\}$ is the direct sum

$$\bigoplus_{\substack{1\leq i\leq m,\\ \nu_p(b_i)<\nu_p(f)}} \mathbb{Z}_p \cdot f/b_i \cdot a_i \oplus \bigoplus_{\substack{1\leq i\leq m,\\ \nu_p(b_i)\geq\nu_p(f)}} \mathbb{Z}_p \cdot a_i.$$

Note that $\vartheta_n(Z^n(G,D))$ equals $\bigoplus_{1\leq i\leq m, b_i=0} \mathbb{Z}_p \cdot a_i$, and hence there exists a $\mathbb{Z}_p$-submodule K of $Z^n(G,D/f\cdot D)$ such that $Z^n(G,D/f\cdot D)$ is the direct sum $\vartheta_n(Z^n(G,D)) \oplus K$. By applying Lemma 8 to $Z^n(G,D/f\cdot D)$ we obtain that $K \cong H^{n+1}(G,f\cdot D) = H^{n+1}(G,A)$. Further, we have $\operatorname{Im}\vartheta_n \cap K = \vartheta_n(Z^n(G,D)) \cap K = 0$. The result for (i) follows.

ad (ii): This statement is a direct consequence of (i). □

A right inverse of the connecting homomorphism is described in

Corollary 10. *Let A, B and ϑ_n be as in Lemma 9. Further, let $\gamma_1,\ldots,\gamma_a \in Z^{n+1}(G,A)$ be such that $H^{n+1}(G,A) = \bigoplus\langle\gamma_i + B^{n+1}(G,A)\rangle$ and let $\psi_1,\ldots,\psi_a \in C^n(G,B)$ satisfy $d^n(\psi_i) = \gamma_i$. Assume that $\vartheta \circ \psi_i \in Z^n(G,B/A)$ has the same order as $\gamma_i + B^{n+1}(G,A)$. Define the group homomorphism ϖ via*

$$\varpi : H^{n+1}(G,A) \to Z^n(G,B/A), \quad \gamma_i + B^{n+1}(G,A) \mapsto \vartheta \circ \psi_i.$$

Then ϖ is a well-defined group homomorphism and the composition $\delta_n \circ \varpi$ is the identity map on $H^{n+1}(G,A)$, where $\delta_n : Z^n(G,B/A) \to H^{n+1}(G,A)$ denotes the connecting homomorphism.

Proof. Since $d^n(\psi_i)$ lies in $Z^{n+1}(G,A)$ we have $\vartheta \circ d^n(\psi_i) = 0$ and hence $\vartheta \circ \psi_i \in Z^n(G,B/A)$. It follows that the group homomorphism ϖ is well-defined. By construction of the connecting homomorphism we have $\delta_n(\vartheta\circ\psi_i) = d^n(\psi_i) + B^{n+1}(G,A) = \gamma_i + B^{n+1}(G,A)$. □

The maps ψ_i in Corollary 10 can be constructed as follows: let o_i be the order of $\gamma_i + B^{n+1}(G,A) \in H^{n+1}(G,A)$. Then there exists $\psi_i' \in C^n(G,A)$ with $d^n(\psi_i') = o_i \cdot \gamma_i$. Since o_i is at most $f =$

$\exp H^{n+1}(G,A)$ and A is a submodule of $f \cdot B$, we may define $\psi_i := o_i^{-1} \cdot \psi_i' \in C^n(G,B)$. It follows that $d^n(\psi_i) = \gamma_i$ and $\vartheta \circ \psi_i$ has order o_i.
Let us consider Lemma 9 for $n = 0$. By definition, the group of 0-coboundaries is trivial and hence $H^0(G,M) = Z^0(G,M)$ for every G-module M. Further, note that $H^0(G,M)$ equals $C_M(G) := \{m \in M \mid m.g = m \text{ for all } m \in M\}$. Lemma 9 yields

Corollary 11. *Denote $f := \exp H^1(G,A)$. Assume that B is torsion-free, $A \leq f \cdot B$ and A is a free $\mathbb{Z}_p$-module of finite rank. Let ϑ_0 be the group homomorphism $C_B(G) \to C_{B/A}(G)$, $b \mapsto b + A$. Then there is a short exact sequence*

$$0 \to \operatorname{Im} \vartheta_0 \hookrightarrow C_{B/A}(G) \to H^1(G,A) \to 0,$$

which splits.

Chapter 3

Automorphism groups of group extensions

Robinson has described the automorphism groups of group extensions in [25]. In this section we summarize the main results. Let P be a group and T be a P-module. For a cocycle $\gamma \in Z^2(P,T)$ let $E(\gamma)$ denote the extension of T by P via γ; that is, $E(\gamma)$ is the product set $P \times T$ endowed with the group multiplication $(g,t)(h,s) = (gh, t.h + s + \gamma(g,h))$.
The group extension $E(\gamma)$ induces a short exact sequence

$$0 \longrightarrow T \longrightarrow E(\gamma) \longrightarrow P \longrightarrow 1$$

consisting of an embedding $T \hookrightarrow E(\gamma)$, $t \mapsto (1,t)$ and a projection $E(\gamma) \twoheadrightarrow P$, $(g,t) \mapsto g$. Let α be an automorphism of $E(\gamma)$ such that α fixes T setwise. Then there exist unique automorphisms $\beta \in \operatorname{Aut} P$ and $\epsilon \in \operatorname{Aut} T$ such that Diagram 1 commutes. It is tempting to suggest that for every $\beta \in \operatorname{Aut} P$ and $\epsilon \in \operatorname{Aut} T$ there is an automorphism α leading to a commutative Diagram 1. This is in general not the case.

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\epsilon"] & E(\gamma) \arrow[r] \arrow[d, "\alpha"] & P \arrow[r] \arrow[d, "\beta"] & 1 \\
0 \arrow[r] & T \arrow[r] & E(\gamma) \arrow[r] & P \arrow[r] & 1
\end{tikzcd}$$

Diagram 1

Definition 12. Let $\beta \in \operatorname{Aut} P$ and $\epsilon \in \operatorname{Aut} T$. We say that (β, ϵ) is inducible to $E(\gamma)$ if there is $\alpha \in \operatorname{Aut} E(\gamma)$ such that Diagram 1 commutes. The automorphism α is called a lifting of (β, ϵ).

We recall the concept of *compatible pairs* to decide whether a pair $(\beta, \epsilon) \in \operatorname{Aut} P \times \operatorname{Aut} T$ is inducible or not. Let $\bar{\cdot}$ denote the homomorphism $P \to \operatorname{Aut} T, g \mapsto \overline{g}$ induced by the group action of P on T. Then the set

$$\operatorname{Comp}(P,T) := \{(\beta, \epsilon) \in \operatorname{Aut} P \times \operatorname{Aut} T \mid \overline{g^\beta} = \overline{g}^\epsilon \text{ for all } g \in P\}$$

is a subgroup of the direct product $\operatorname{Aut} P \times \operatorname{Aut} T$. We call the elements of $\operatorname{Comp}(P,T)$ compatible pairs of P and T. The automorphism group $\operatorname{Aut} T$ acts naturally on the abelian group T via $t.\epsilon := \epsilon(t)$ for $t \in T$ and $\epsilon \in \operatorname{Aut} T$. Further, for $n \in \mathbb{N}$ the group action of $\operatorname{Comp}(P,T)$ on $Z^n(P,T)$ is defined by $\gamma^{(\beta,\epsilon)}(g_1,\dots,g_n) = \gamma({g_1}^{\beta^{-1}},\dots,{g_n}^{\beta^{-1}}).\epsilon$ for $\gamma \in Z^n(P,T)$ and $(\beta,\epsilon) \in \operatorname{Comp}(P,T)$. This induces an action of $\operatorname{Comp}(P,T)$ on $H^n(P,T)$.
For a map $\psi : P \to T$ we write $\alpha(\psi)$ to denote the map

$$\alpha(\psi) : P \times P \to T,\ (g,h) \mapsto \psi(g).h + \psi(h) - \psi(gh).$$

Recall that $\mathrm{C}^1(P,T)$ denote the set of maps $\psi : P \to T$ with $\psi(1) = 0$ and that the group of 2-coboundaries $B^2(P,T)$ consists of $\alpha(\psi)$ with $\psi \in C^1(P,T)$.

Theorem 13 (cf. [25, Theorem (4.1)])**.** *Let $\gamma \in Z^2(P,T)$ and $(\beta,\epsilon) \in \operatorname{Aut} P \times \operatorname{Aut} T$.*

(i) *The pair (β,ϵ) is inducible to $E(\gamma)$ if and only if $(\beta,\epsilon) \in \operatorname{Stab}_{\operatorname{Comp}(P,T)}(\gamma + B^2(P,T))$.*

Assume that (β,ϵ) is an inducible pair and let $\psi \in C^1(P,T)$ be such that $\alpha(\psi) = \gamma^{(\beta,\epsilon)} - \gamma$. Then

(ii) *the map $\varphi(\beta,\epsilon,\psi)$ defined by*

$$\varphi(\beta,\epsilon,\psi) : E(\gamma) \to E(\gamma),\ (g,t) \mapsto (g^\beta, \psi(g^\beta) + t.\epsilon)$$

is a well-defined automorphism of $E(\gamma)$ and a lifting of (β,ϵ),

(iii) *there are exactly $|Z^1(P,T)|$ liftings of (β,ϵ), namely $\{\varphi(\beta,\epsilon,\psi+\varrho) \mid \varrho \in Z^1(P,T)\}$.*

A proof of Theorem 13 can be found in [25]. Since our notation differs from the one in [25] and because of the importance of Theorem 13 in this thesis, we give a proof.

Proof. It can easily be verified that every possible lifting of (β,ϵ) has the form $\varphi(\beta,\epsilon,\psi)$ with $\psi \in C^1(P,T)$. Thus it suffices to show that for $\psi \in C^1(P,T)$ the map $\varphi := \varphi(\beta,\epsilon,\psi)$ is a well-defined automorphism if and only if (β,ϵ) is a compatible pair and $\alpha(\psi) = \gamma^{(\beta,\epsilon)} - \gamma$. Let $(g,t),(g',t') \in E(\gamma)$. Then the product $\varphi(g,t) \cdot \varphi(g',t')$ equals

$$(g^\beta, \psi(g^\beta) + t.\epsilon) \cdot (g'^\beta, \psi'(g'^\beta) + t'.\epsilon) = ((gg')^\beta, (t.\epsilon).g'^\beta + t'.\epsilon + \psi(g^\beta).g'^\beta + \psi(g'^\beta) + \gamma(g^\beta, g'^\beta)).$$

On the other hand, the image of $(g,t)\cdot(g',t') = (gg', \gamma(g,g')+t.g'+t')$ under φ is $((gg')^\beta, \psi((gg')^\beta)+\gamma(g,g').\epsilon + (t.g').\epsilon + t'.\epsilon)$. It follows that $\varphi(g,t)\cdot\varphi(g',t')$ equals $\varphi((g,t)\cdot(g',t'))$ if and only if

$$(t.\epsilon.g'^\beta - t.g'.\epsilon) + \psi(g^\beta).g'^\beta + \psi(g'^\beta) - \psi((gg')^\beta) + \gamma(g^\beta, g'^\beta) - \gamma(g,g').\epsilon = 0 \tag{3.0.1}$$

holds. For $(g,t) = (1,0)$ Equation (3.0.1) yields $t.\epsilon.g'^\beta = t.g'.\epsilon$. Thus, if φ is a group homomorphism then (β,ϵ) is a compatible pair. For a compatible pair (β,ϵ) Equation (3.0.1) simplifies to

$$\psi(g^\beta).(g'^\beta) + \psi(g'^\beta) - \psi((gg')^\beta) = \gamma(g,g').\epsilon - \gamma(g^\beta, g'^\beta)$$

It follows that φ is a homomorphism if and only if (β,ϵ) is a compatible pair and ψ satisfies $\alpha(\psi) = \gamma^{(\beta,\epsilon)} - \gamma$. □

For $\gamma \in Z^2(P,T)$ let $\operatorname{Aut}\gamma = \operatorname{Aut}(\gamma, P, T)$ be defined as

$$\operatorname{Aut}\gamma := \{(\beta,\epsilon,\psi) \mid (\beta,\epsilon) \text{ is inducible to } E(\gamma) \text{ and } \psi \in \mathrm{C}^1(P,T) \text{ with } \alpha(\psi) = \gamma^{(\beta,\epsilon)} - \gamma\}.$$

Assume that T embeds as a characteristic subgroup in $E(\gamma)$. Then every automorphism α induces canonically automorphisms α_P and α_T on $P \cong E(\gamma)/T$ and T, respectively. In particular the map $\nu : \operatorname{Aut} E(\gamma) \to \operatorname{Aut} P \times \operatorname{Aut} T$, $\alpha \mapsto (\alpha_P, \alpha_T)$ is a well-defined homomorphism. Hence every automorphism α is a lifting of an inducible pair, namely (α_P, α_T). This enables us to define a 1-1 correspondence between $\operatorname{Aut}\gamma$ and $\operatorname{Aut} E(\gamma)$.

Lemma 14. *As in the preceding paragraph, let T embed as a characteristic subgroup in $E(\gamma)$. Then the map*

$$\varphi : \operatorname{Aut}\gamma \to \operatorname{Aut} E(\gamma),\ \ (\beta,\epsilon,\psi) \mapsto \varphi(\beta,\epsilon,\psi)$$

is a well-defined bijection.

The bijection of Lemma 14 induces a group structure on $\operatorname{Aut}\gamma$ with the group multiplication

$$(\beta_1,\epsilon_1,\psi_1)(\beta_2,\epsilon_2,\psi_2) = (\beta_1\beta_2, \epsilon_1\epsilon_2, \psi_1^{(\beta_2,\epsilon_2)} + \psi_2), \tag{3.0.2}$$

where $\psi_1^{(\beta_2,\epsilon_2)}$ is defined as $P \to T$, $g \mapsto \psi_1(g^{\beta_2^{-1}}).\epsilon_2$. Then φ can be considered as a group isomorphism. Further, note that the composition $\nu \circ \varphi : \operatorname{Aut}\gamma \mapsto \operatorname{Aut} P \times \operatorname{Aut} T$ induces the short exact sequence

$$0 \to Z^1(P,T) \to \operatorname{Aut}\gamma \to \operatorname{Stab}_{\operatorname{Comp}(P,T)}(\gamma + B^2(P,T)) \to 1. \tag{3.0.3}$$

Chapter 4

Polycyclic groups

In this chapter we consider polycyclic groups. For further details on polycyclic groups we refer to [28] and we recommend [12] and [6] for those readers, who are interested in a computational approach. First we give the basic definitions and notations of polycyclic groups in Section 4.1, here we follow [6]. Then we focus on group extensions of abelian groups by polycyclic groups in Section 4.2. Of particular interest are those extensions which arise from polycyclic groups: they are also polycyclic, and presentations of these groups can be deduced from corresponding 2-cocycles. This fact is more or less folk-lore and leads to a map from 2-cocycles to group presentations. We additionally construct a right inverse to this map.

Finally, in Subsection 4.2.2 we recall some results of [6, Section 6.3], in which representations of the first and second cohomology group are described and discussed. Although we follow [6, Section 6.3], we have chosen a different approach to these representations by giving equivalent but less explicit definitions for them. The reason for this is twofold: we hope that the representations appear more natural and the proofs less technical to the reader.

4.1 Definitions and basic features

A group G is called *polycyclic*, if there is a *polycyclic series* of G; that is, a subnormal series $G = G(1) \geq G(2) \geq \cdots \geq G(m+1) = 1$ of finite length with nontrivial cyclic factors. Obviously, the class of finite polycyclic groups coincides with the class of finite solvable groups. In particular every finite p-group is polycyclic. However not every solvable group is polycyclic.

Theorem 15 ([28, Proposition 1.3]). *Let G be a group. Then G is polycyclic if and only if G is solvable and each subgroup of G is finitely generated.*

The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is an example for a solvable, nonpolycyclic group: the derived subgroup of the wreath product lies in the abelian base of $\mathbb{Z} \wr \mathbb{Z}$ and thus $\mathbb{Z} \wr \mathbb{Z}$ is solvable. Since the base is not finitely generated, the group $\mathbb{Z} \wr \mathbb{Z}$ is not polycyclic by Theorem 15.

The algorithmic approach to polycyclic groups is well-developed, for example there are efficient algorithms to compute group presentations and group extensions of polycyclic groups, see [12]. These algorithms use the concept of polycyclic sequences.

Definition 16. Let G be a group. Then a *polycyclic sequence* of G is a finite sequence $\mathcal{G} = (g_1, \ldots, g_m)$ of elements of G such that the series $(G(i) \mid 1 \leq i \leq m+1)$ defined by $G(m+1) = 1$ and $G(i) = \langle g_i, G(i+1) \rangle$ for $1 \leq i \leq m$ is a polycyclic series of G.

Evidently, a group G is polycyclic if and only if G has a polycyclic sequence [6, Lemma 1.3].
Let G be a polycyclic group and let $\mathcal{G} = (g_1, \ldots, g_m)$ denote a polycyclic sequence of G. Then the *relative orders* of $\mathcal{G}$ are defined as $r_i := [G(i) : G(i+1)] \in \mathbb{N} \cup \{\infty\}$. Note that for $g \in G(i)$ there exists a unique integer e_i such that $g \cdot G(i+1)$ equals $g_i^{e_i} \cdot G(i+1)$ and $0 \leq e_i < r_i$ if $r_i < \infty$. Hence the polycyclic sequence induces an injection

$$\exp_{\mathcal{G}} :\ G \to \mathbb{Z}^m, \quad g \mapsto (e_1, \ldots, e_m),$$

where $g = g_1^{e_1} \cdots g_m^{e_m}$ and $0 \leq e_i < r_i$ for i with $r_i < \infty$. We call $\exp_{\mathcal{G}}(g)$ the *exponent vector* of g with respect to the polycyclic sequence $\mathcal{G}$. For $1 \leq i \leq m$ we write $\exp_{\mathcal{G}}(g)_i$ to denote the i-th entry of $\exp_{\mathcal{G}}(g)$. The exponent vector allows us to define the *depth* and *leading exponent* of an element:

$$\begin{array}{rccl} \mathrm{dep}_{\mathcal{G}} : & G & \to & \{1, \ldots, m+1\}, \\ & g & \mapsto & \begin{cases} f \text{ with } g \in G(f) \setminus G(f+1), & \text{if } g \neq 1, \\ m+1, & \text{otherwise}, \end{cases} \\ \mathrm{lead}_{\mathcal{G}} : & G \setminus 1 & \to & \mathbb{Z}, \\ & g & \mapsto & \exp_{\mathcal{G}}(g)_f \text{ with } f = \mathrm{dep}_{\mathcal{G}}(g). \end{array}$$

The polycyclic sequence $\mathcal{G}$ induces a group presentation of G. First, note that there are integers $a_{i,j,k}$ such that $0 \leq a_{i,j,k} < r_k$ for k with $r_k < \infty$, and

$$g_i^{r_i} = (g_{i+1}^{a_{i,i,i+1}} \cdots g_m^{a_{i,i,m}})^{-1}, \quad \text{for } 1 \leq i \leq m \text{ with } r_i < \infty, \tag{4.1.1}$$

$$g_j^{g_i} = (g_{i+1}^{a_{i,j,i+1}} \cdots g_m^{a_{i,j,m}})^{-1}, \quad \text{for } 1 \leq i < j \leq m, \tag{4.1.2}$$

$$g_j^{g_i^{-1}} = (g_{i+1}^{a_{j,i,i+1}} \cdots g_m^{a_{j,i,m}})^{-1}, \quad \text{for } 1 \leq i < j \leq m; \tag{4.1.3}$$

in particular $a_{i,j,k}$ are entries of exponent vectors. Let $F(x_1, \ldots, x_m)$ be the free group on the set $\{x_1, \ldots, x_m\}$ and let $\Omega(\mathcal{G})$ be the set consisting of the relators

$$\begin{array}{lll} \omega_{i,i} & := x_i^{r_i} x_{i+1}^{a_{i,i,i+1}} \cdots x_m^{a_{i,i,m}}, & \text{for } 1 \leq i \leq m \text{ with } r_i < \infty, \\ \omega_{i,j} & := x_i^{-1} x_j x_i x_{i+1}^{a_{i,j,i+1}} \cdots x_m^{a_{i,j,m}}, & \text{for } 1 \leq i < j \leq m, \\ \omega_{j,i} & := x_i x_j x_i^{-1} x_{i+1}^{a_{j,i,i+1}} \cdots x_m^{a_{j,i,m}}, & \text{for } 1 \leq i < j \leq m \text{ with } r_i = \infty. \end{array}$$

The presentation $\mathrm{pc}(\mathcal{G}) := \langle x_1, \ldots, x_m \mid \Omega(\mathcal{G}) \rangle$ is called the *pc-presentation* of G with respect to the polycyclic sequence $\mathcal{G}$.

Theorem 17 ([12, Theorem 8.8])**.** *Let G be a polycyclic group with polycyclic series $\mathcal{G}$. Then $pc(\mathcal{G})$ is a group presentation of G.*

The relators $\omega_{i,i}$ and $\omega_{i,j}$ with $i \neq j$ are called *power-relators* and *commutator-relators*, respectively. We call the commutator-relators $\omega_{i,j}$ yielding $[g_i, g_j] = 1$ trivial. Quite often a shorten notation of a pc-presentation $\langle x_1, \ldots, x_m \mid \Omega(\mathcal{G}) \rangle$ is used, in which the trivial commutator-relators are omitted: let R be a subset of $\Omega(\mathcal{G})$ such that the complement of R in $\Omega(\mathcal{G})$ consists only of trivial commutator-relators. Then we write $\mathrm{pc}\langle x_1, \ldots, x_m \mid R \rangle$ to denote $\mathrm{pc}(\mathcal{G}) = \langle x_1, \ldots, x_m \mid \Omega(\mathcal{G}) \rangle$.
Next, we consider subgroups of polycyclic groups. As a direct consequence of Theorem 15 we obtain

Theorem 18. *The class of polycyclic groups is closed under taking subgroups, intersections, factor groups and group extensions.*

Let U be a subgroup of G and recall that $G = G(1) > G(2) > \cdots > G(m+1) = 1$ is the polycyclic series induced by the polycyclic sequence $\mathcal{G}$. Then the series

$$U \cap G = U \cap G(1) \geq U \cap G(2) \geq \cdots \geq U \cap G(m+1) = 1$$

has cyclic factors $U \cap G(i)/U \cap G(i+1) \cong (U \cap G(i)) \cdot G(i+1)/G(i+1)$. Thus there exists a polycyclic sequence $\mathcal{U} = (u_i \mid i \in I)$ of U with $I \subseteq \{1, \dots, m\}$ and $u_i \in G(i) \setminus G(i+1)$. Such a sequence is called an *induced polycyclic sequence* of U with respect to $\mathcal{G}$.

4.2 Cohomology groups of polycyclic groups

This section deals with group extensions of abelian groups by polycyclic groups. For this purpose, let G denote a polycyclic group with polycyclic sequence $\mathcal{G} = (g_1, \dots, g_m)$ and corresponding relative orders $r_1, \dots, r_m$. Further, let T be a G-module. For $g \in G$ and $t \in T$ we write $t.g$ to denote the image of t under the action of g. In Subsection 4.2.1 we construct maps between $Z^2(G,T)$ and pc-presentations of group extensions of T by G for the case that T is polycyclic; and in Subsection 4.2.2 we consider faithful representations of the first and second cohomology group. Recall that for a cocycle $\gamma \in Z^2(G,T)$ we write $E(\gamma)$ to denote the group extension defined by γ, that is, $E(\gamma)$ is the product set $G \times T$ with the group multiplication $(g,t)(h,s) = (gh, t.h + s + \gamma(g,h))$. Let

$$F := F(x_1, \dots, x_m)$$

be the free group on the set $X := \{x_1, \dots, x_m\}$ and for $\omega \in F$ let $\omega(\mathcal{G})$ denote the image of ω under the homomorphism $F \to G$ defined by $x_i \mapsto g_i$. Further, let $\tau_{E(\gamma)} : F \to E(\gamma)$ be the group homomorphism with $\tau_{E(\gamma)}(x_i) = (g_i, 0)$, and define

$$\delta_\gamma : F \to T,\ \omega \mapsto \delta_\gamma(\omega) \text{ with } \tau_{E(\gamma)}(\omega) = (\omega(\mathcal{G}), \delta_\gamma(\omega)).$$

We write X^{-1} to denote $\{x_1^{-1}, \dots, x_m^{-1}\} \subseteq F$. The image of δ_γ is described explicitly in

Lemma 19. *Let $\omega = r_1 \cdots r_l \in F$ be such that $l \in \mathbb{N}$ and $r_i \in X \cup X^{-1}$. For $1 \leq i \leq l$ denote $s_i := r_i(\mathcal{G})$, $v_i := (r_i \cdots r_l)(\mathcal{G})$ and put $v_{l+1} := 1$. Then for $\gamma \in Z^2(G,T)$, we have*

$$\delta_\gamma(\omega) = -\sum_{\substack{1 \leq i \leq l, \\ r_i \in X^{-1}}} \gamma(s_i^{-1}, s_i).v_{i+1} + \sum_{1 \leq i \leq l-1} \gamma(s_i, v_{i+1}).$$

Proof. For a word ω let $\nu(\omega)$ be the expression on the right hand side of the equation above. Now fix a word $\omega = r_1 \cdots r_l$ of length $l \in \mathbb{N}$, let ω_i denote the tail subword $r_i \cdots r_l$ and put $\omega_{l+1} := 1$. For $g \in G$ the inverse of $(g,0) \in E(\gamma)$ is $(g^{-1}, -\gamma(g, g^{-1}))$ and hence we have

$$\delta_\gamma(r_i) = \nu(r_i) = \begin{cases} 0, & \text{if } r_i \in \mathcal{G}, \\ -\gamma(s_i^{-1}, s_i), & \text{if } r_i \in \mathcal{G}^{-1}. \end{cases}$$

Further, the image of ω_i under $\tau_{E(\gamma)}$ equals $\tau_{E(\gamma)}(r_i) \cdot \tau_{E(\gamma)}(\omega_{i+1}) = (v_i, \delta_\gamma(r_i).v_{i+1} + \delta_\gamma(\omega_{i+1}))$ and thus $\delta_\gamma(\omega_i) = \delta_\gamma(r_i).v_{i+1} + \delta_\gamma(\omega_{i+1})$. Since the recursion formula $\nu(\omega_i) = \nu(r_i).v_{i+1} + \nu(\omega_{i+1})$ also holds, it follows that $\nu(\omega) = \delta_\gamma(\omega)$. □

4.2.1 From 2-cocycles to pc-presentations and vice versa

In this subsection we assume that the abelian group T is polycyclic, in other words T is finitely generated. First, we show how to write down a pc-presentation of an extension $0 \to T \to E \to G \to 1$ by modifying and extending pc-presentations of G and T. Let $\mathcal{G} = (g_1, \ldots, g_m)$ and $\mathcal{T} = (t_1, \ldots, t_d)$ be polycyclic sequences of G and T, respectively, and let $r_1, \ldots, r_m$ be the relative orders of $\mathcal{G}$.

Definition 20. Let $0 \to T \xrightarrow{\iota_E} E \xrightarrow{\nu_E} G \to 1$ be a group extension of T by G. A polycyclic sequence $\mathcal{E}$ of E is compatible with $(\mathcal{G}, \mathcal{T})$ w.r.t. (ν_E, ι_E) if $\mathcal{E}$ is of the form $(\tilde{g}_1, \ldots, \tilde{g}_m, \tilde{t}_1, \ldots, \tilde{t}_d)$ with $\nu_E(\tilde{g}_i) = g_i$ and $\iota_E(t_i) = \tilde{t}_i$.

Let
$$0 \to T \xrightarrow{\iota_E} E \xrightarrow{\nu_E} G \to 1$$
be a group extension and let $\mathcal{E} = (\tilde{g_1}, \ldots, \tilde{g_m}, \tilde{t_1}, \ldots, \tilde{t_d})$ be a polycyclic sequence of E which is compatible with $(\mathcal{G}, \mathcal{T})$ w.r.t. (ν_E, ι_E). Our aim is to construct a pc-presentation of E with respect to $\mathcal{E}$. For this purpose, put $X := \{x_1, \ldots, x_m\}$ and $Y := \{y_1, \ldots, y_m\}$, let
$$\langle X \mid \Omega(\mathcal{G})\rangle \text{ and } \langle Y \mid \Omega(\mathcal{T})\rangle$$
be the pc-presentations of G and T, respectively, and let $F(X, Y)$ denote the free group with generating set $X \cup Y$. Further, let $b_{i,j,k}$ and $c_{i,j,k}$ be integers with $\exp_{\mathcal{T}}(-t_j . g_i) = (b_{i,j,1}, \ldots, b_{i,j,d})$ and $\exp_{\mathcal{T}}(-t_j . g_i^{-1}) = (c_{i,j,1}, \ldots, c_{i,j,d})$. This yields $t_j . g_i = -\sum_{k=1}^{d} b_{i,j,k} \cdot t_k$ and $t_j . g_i^{-1} = -\sum_{k=1}^{d} c_{i,j,k} \cdot t_k$. We write $\Omega(\mathcal{G}, \mathcal{T})$ to denote the set of the following relators
$$\begin{array}{ll} x_i^{-1} \cdot y_j \cdot x_i \cdot y_1^{b_{i,j,1}} \cdots y_d^{b_{i,j,d}}, & \text{for } 1 \le i \le m \text{ and } 1 \le j \le d, \\ x_i \cdot y_j \cdot x_i^{-1} \cdot y_1^{c_{i,j,1}} \cdots y_d^{c_{i,j,d}}, & \text{for } 1 \le i \le m \text{ and } 1 \le j \le d \text{ with } r_i = \infty \end{array}$$
in the free group $F(X, Y)$. These relators decode the action of G on T. Now, we modify $\Omega(\mathcal{G})$ to obtain a presentation of E. Let $\tau_{\mathcal{E}} : F(X) \to E$ be the homomorphism defined by $\tau_{\mathcal{E}}(x_i) = \tilde{g}_i$ and for $\omega \in \Omega(\mathcal{G})$ denote
$$\omega_{\mathcal{E}} := y_1^{b_{\omega,1}} \cdots y_d^{b_{\omega,d}} \text{ with } (b_{\omega,1}, \ldots, b_{\omega,d}) = -\exp_{\mathcal{T}}(\iota_E^{-1}(\tau_{\mathcal{E}}(\omega))).$$
Further, let $\Omega(\mathcal{E}, \mathcal{G})$ be the set consisting of the relators $\omega \cdot \omega_{\mathcal{E}}$ with $\omega \in \Omega(\mathcal{G})$.

Lemma 21 (cf. [17, §20 Theorem 1])**.** *Let $\mathcal{E}$ be a polycyclic sequence of E which is compatible with $(\mathcal{G}, \mathcal{T})$, and let $pc(\mathcal{E}) = \langle X, Y \mid \Omega(\mathcal{E})\rangle$ be the pc-presentation of E w.r.t. $\mathcal{E}$. Then $\Omega(\mathcal{E})$ is the union $\Omega(\mathcal{E}, \mathcal{G}) \cup \Omega(\mathcal{T}) \cup \Omega(\mathcal{G}, \mathcal{T})$.*

For a 2-cocycle $\gamma \in Z^2(G, T)$ let $E(\gamma)$ denote the group extension defined by γ. Then $\mathcal{E}(\gamma) = ((g_1, 0), \ldots, (g_m, 0), (1, t_1), \ldots, (1, t_d))$ is a polycyclic sequence of $E(\gamma)$ which is compatible to $(\mathcal{G}, \mathcal{T})$ w.r.t. the natural embedding $T \hookrightarrow E(\gamma)$ and the natural homomorphism $E(\gamma) \twoheadrightarrow G$. Define
$$\mathrm{pc}(\gamma) := \mathrm{pc}(\mathcal{E}(\gamma)).$$

Remark 22. *Lemma 21 enables us to write down an explicit pc-presentation of $pc(\gamma) = \langle X, Y \mid \Omega(\mathcal{E}(\gamma))\rangle$. The set of relators $\Omega(\mathcal{E}(\gamma))$ is $\Omega(\mathcal{E}(\gamma), \mathcal{G}) \cup \Omega(\mathcal{T}) \cup \Omega(\mathcal{G}, \mathcal{T})$ where only the first set $\Omega(\mathcal{E}(\gamma), \mathcal{G})$ depends on the chosen 2-cocycle γ. Recall that the elements of $\Omega(\mathcal{E}(\gamma), \mathcal{G})$ are $\omega \cdot \omega_{\mathcal{E}(\gamma)}$ with $\omega \in \Omega(\mathcal{G}) \le F(X)$. Let $\delta_\gamma(\omega)$ be defined as in the beginning of Section 4.2, and note that the word $\omega_{\mathcal{E}(\gamma)}$ equals $y_1^{b_{\omega,1}} \cdots y_d^{b_{\omega,d}}$ with $(b_{\omega,1}, \ldots, b_{\omega,d}) = -\exp_{\mathcal{T}}(\delta_\gamma(\omega))$. Thus the isomorphism class of $E(\gamma)$ is determined by the images $\delta_\gamma(\omega)$ with $\omega \in \Omega(\mathcal{G})$. A similar result is shown in Lemma 25.*

In what follows we give a right inverse to $Z^2(G,T) \to \{\mathrm{pc}(\mathcal{E}) \mid \mathcal{E} \text{ is compatible with } (\mathcal{G},\mathcal{T})\}$, $\gamma \mapsto \mathrm{pc}(\gamma)$. Let $\mathcal{E}$ be a polycyclic sequence which is compatible with $(\mathcal{G},\mathcal{T})$, and let $\mathrm{pc}(\mathcal{E}) = \langle X, Y \mid \Omega(\mathcal{E})\rangle$ be the pc-presentation of E w.r.t. $\mathcal{E}$. Further let N denote $\langle\langle\Omega(\mathcal{E})\rangle\rangle$, the normal closure of $\Omega(\mathcal{E})$ in $F(X,Y)$, and define the following group homomorphisms

$$\iota: \; T \to \mathrm{pc}(\mathcal{E}), \; t_i \mapsto y_i \cdot N \quad \text{and} \quad \nu: \; \mathrm{pc}(\mathcal{E}) \to G, \; \begin{array}{rcl} y_i \cdot N & \mapsto & 1, \\ x_j \cdot N & \mapsto & g_j. \end{array}$$

For an element $g \in G$, the word $\mathrm{norm}(g) \in F(X)$ is defined as $x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}$, where $(e_1,\ldots,e_m) = \exp_{\mathcal{G}}(g)$. We call $\mathrm{norm}(g)$ the *normal form* of g with respect to $\mathcal{G}$. Evidently, for g, $h \in G$ the element $\mathrm{norm}(gh)^{-1} \cdot \mathrm{norm}(g) \cdot \mathrm{norm}(h)$ lies in the normal closure of $\Omega(\mathcal{G})$ in $F(X)$. This enables us to define

$$\gamma_{\mathcal{E}} : G \times G \to T, \; (g,h) \mapsto \iota^{-1}(\mathrm{norm}(gh)^{-1} \cdot \mathrm{norm}(g) \cdot \mathrm{norm}(h) \cdot N),$$

where $\iota^{-1}(\omega \cdot N)$ denote the preimage of an element $\omega \cdot N \in \mathrm{Im}\,\iota$. Recall that for a word $\omega \in \Omega(\mathcal{G})$ there exists a unique word $\omega_{\mathcal{E}} \in F(Y)$ with $\omega \cdot \omega_{\mathcal{E}} \in \Omega(\mathcal{E})$ by Lemma 21.

Lemma 23. *Let $\omega = \omega_1^{s_1} \cdots \omega_n^{s_n} \in F(X,Y)$ with $n \in \mathbb{N}$, $\omega_i \in \Omega(\mathcal{G})$ and $s_i \in F(X,Y)$. Then*

(i) *$\omega \cdot N \in \mathrm{Im}\,\iota$ and $\iota^{-1}(\omega \cdot N) = -\sum_{1 \le i \le n} \iota^{-1}((\omega_i)_{\mathcal{E}} \cdot N).\nu(s_i)$,*

(ii) *the map $\gamma_{\mathcal{E}}$ is a well-defined 2-cocycle and*

(iii) *$pc(\gamma_{\mathcal{E}}) = pc(\mathcal{E})$.*

Proof. Denote $E := \mathrm{pc}(\mathcal{E})$.

ad (i): Since $\mathrm{Im}\,\iota$ is a group and $\iota^{-1} : \mathrm{Im}\,\iota \to T$ is a group isomorphism, it suffices to show (i) for $\omega = \tilde{\omega}^s$ with $\tilde{\omega} \in \Omega(\mathcal{G})$ and $s \in F(X,Y)$. As $\tilde{\omega} \cdot \tilde{\omega}_{\mathcal{E}}$ is an element of $N = \langle\langle\Omega(\mathcal{E})\rangle\rangle$, we have $\tilde{\omega}^s \cdot N = (\tilde{\omega}_{\mathcal{E}}^{-1})^s \cdot N$. Further, the element $\tilde{\omega}_{\mathcal{E}}^{-1} \cdot N$ lies in the image $\mathrm{Im}\,\iota$ and thus $\tilde{\omega}^s \cdot N \in \mathrm{Im}\,\iota$. Since $0 \to T \xrightarrow{\iota} E \xrightarrow{\nu} G \to 1$ is a group extension of T by G, it follows that $\iota^{-1}((\tilde{\omega}_{\mathcal{E}}^{-1})^s \cdot N)$ equals $-\iota^{-1}(\tilde{\omega}_{\mathcal{E}} \cdot N).\nu(s)$.
ad (ii): Note that $\{\mathrm{norm}(g) \mid g \in G\}$ is a transversal of $E/\mathrm{Im}\,\iota$. Then it follows by group extension theory (see for example [17, p.154]) that $\gamma_{\mathcal{E}}$ is a well-defined 2-cocycle.
ad (iii): Denote $\gamma := \gamma_{\mathcal{E}}$. By Lemma 21 the set $\Omega(\mathcal{E})$ equals $\Omega(\mathcal{E},\mathcal{G}) \cup \Omega(\mathcal{G},\mathcal{T}) \cup \Omega(\mathcal{T})$. Then by Remark 22 it suffices to show $\delta_\gamma(\omega) = -\iota^{-1}(\omega_{\mathcal{E}} \cdot N)$ for $\omega \in \Omega(\mathcal{G})$ in order to prove $\mathrm{pc}(\gamma) = E$. First, note that for $g \in G \setminus 1$ and $l = \mathrm{dep}_{\mathcal{G}}(g)$ we have $\mathrm{norm}(g) = \mathrm{norm}(g_l) \cdot \mathrm{norm}(g_l^{-1} g)$. Further, for $\epsilon \in \{-1,+1\}$ and $1 \le k \le l$ with $r_k = \infty$ the equality $\mathrm{norm}(g) = \mathrm{norm}(g_k^\epsilon) \cdot \mathrm{norm}(g_k^{-\epsilon} g)$ also holds. Hence

$$\gamma(g_l, g_l^{-1} g) = 0, \quad \text{if } l = \mathrm{dep}_{\mathcal{G}}(g), \tag{4.2.4}$$

$$\gamma(g_k^\epsilon, g_k^{-\epsilon} g) = 0, \quad \text{if } \epsilon \in \{-1,1\} \text{ and } 1 \le k \le \mathrm{dep}_{\mathcal{G}}(g) \text{ with } r_k = \infty. \tag{4.2.5}$$

In what follows, we show that Lemma 19 and Equations (4.2.4) and (4.2.5) yield $\delta_\gamma(\omega) = -\iota^{-1}(\omega_{\mathcal{E}} \cdot N)$ for every $\omega \in \Omega(\mathcal{G})$. Now, fix $1 \le i \le m$ and consider the case, in which the relative order r_i is finite. Let $\omega_{i,i} = x_i^{r_i} x_{i+1}^{a_{i,i,i+1}} \cdots x_m^{a_{i,i,m}}$ be the power-relator of g_i. Then the definition of γ and (i) yield

$$\begin{array}{rcl} \gamma(g_i, g_i^{-1}) & = & \iota^{-1}(\mathrm{norm}(g_i g_i^{-1})^{-1} \cdot \mathrm{norm}(g_i) \cdot \mathrm{norm}(g_i^{-1}) \cdot N) \\ & = & \iota^{-1}(1 \cdot x_i \cdot x_i^{r_i - 1} x_{i+1}^{a_{i,i,i+1}} \cdots x_m^{a_{i,i,m}} \cdot N) = \iota^{-1}(\omega_{i,i} \cdot N) = -\iota^{-1}((\omega_{i,i})_{\mathcal{E}} \cdot N) \quad \text{and} \\ \gamma(g_i^{-1}, g_i) & = & \iota^{-1}(\mathrm{norm}(g_i^{-1}) \cdot \mathrm{norm}(g_i) \cdot N) = \iota^{-1}((\mathrm{norm}(g_i) \cdot \mathrm{norm}(g_i^{-1}))^{g_i} \cdot N) \\ & = & \gamma(g_i, g_i^{-1}).g_i = -\iota^{-1}((\omega_{i,i})_{\mathcal{E}} \cdot N).g_i \; . \end{array}$$

It follows by Lemma 19, Equations (4.2.4) and (4.2.5) that

$$\delta_\gamma(\omega_{i,i}) = \gamma(g_i, g_i^{r_i-1} g_{i+1}^{a_{i,i,i+1}} \cdot g_m^{a_{i,i,m}}) + 0 + \ldots + 0 = \gamma(g_i, g_i^{-1}) = -\iota^{-1}((\omega_{i,i})_\mathcal{E} \cdot N).$$

Further, for a commutator-relator of G, say $\omega_{i,j} = x_i^{-1} x_j x_i x_{i+1}^{a_{i,j,i+1}} \cdots x_m^{a_{i,j,m}}$ with $i < j$, we have

$$\delta_\gamma(\omega_{i,j}) \quad = \quad -\gamma(g_i, g_i^{-1}).g_i + \gamma(g_i^{-1}, g_i) + \gamma(g_j, g_i g_{i+1}^{a_{i,j,i+1}} \cdots g_m^{a_{i,j,m}}) = \gamma(g_j, g_i g_{i+1}^{a_{i,j,i+1}} \cdots g_m^{a_{i,j,m}}).$$

Since $g_j \cdot g_i g_{i+1}^{a_{i,j,i+1}} \cdots g_m^{a_{i,j,m}}$ equals g_i, the image $\gamma(g_j, g_i g_{i+1}^{a_{i,j,i+1}} \cdots g_m^{a_{i,j,m}})$ is

$$\iota^{-1}(x_i^{-1} \cdot x_j \cdot x_i x_{i+1}^{a_{i,j,i+1}} \cdots x_m^{a_{i,j,m}} \cdot N) = \iota^{-1}(\omega_{i,j} \cdot N) = -\iota^{-1}((\omega_{i,j})_\mathcal{E} \cdot N).$$

It follows that $\delta_\gamma(\omega_{i,j}) = -\iota^{-1}((\omega_{i,j})_\mathcal{E} \cdot N)$.

Now, let us assume that $r_i = \infty$. Then $\gamma(g_i, g_i^{-1})$ and $\gamma(g_i^{-1}, g_i)$ are trivial by Equation (4.2.5). This yields for $i < j$

$$\delta_\gamma(\omega_{i,j}) \quad = \quad \gamma(g_j, g_i g_{i+1}^{a_{i,j,i+1}} \cdots g_m^{a_{i,j,m}}) = \iota^{-1}(\omega_{i,j} \cdot N) = -\iota^{-1}((\omega_{i,j})_\mathcal{E} \cdot N).$$

Further, for the commutator-relator $\omega_{j,i} = x_i x_j x_i^{-1} x_{i+1}^{a_{j,i,i+1}} \cdots x_m^{a_{j,i,m}}$ the image $\delta_\gamma(\omega_{j,i})$ equals

$$\begin{aligned} & \gamma(g_i, g_j g_i^{-1} g_{i+1}^{a_{j,i,i+1}} \cdots g_m^{a_{j,i,m}}) + \gamma(g_j, g_i^{-1} g_{i+1}^{a_{j,i,i+1}} \cdots g_m^{a_{j,i,m}}) \\ = \quad & \iota^{-1}(1^{-1} \cdot x_i \cdot x_i^{-1} \cdot N) + \iota^{-1}((x_i^{-1})^{-1} \cdot x_j \cdot x_i^{-1} x_{i+1}^{a_{j,i,i+1}} \cdots x_m^{a_{j,i,m}} \cdot N) = -\iota^{-1}((\omega_{j,i})_\mathcal{E} \cdot N). \end{aligned}$$

It follows that $\mathrm{pc}(\gamma) = E$.

$\square$

4.2.2 Determining first and second cohomology group

Let again G denote a polycyclic group with polycyclic sequence $\mathcal{G} = (g_1, \ldots, g_m)$, say, and let T be a G-module, which is not necessarily polycyclic. Further, let $\langle x_1, \ldots, x_m \mid \Omega(\mathcal{G}) \rangle$ be the pc-presentation of G with respect to $\mathcal{G}$, denote $\Omega := \Omega(\mathcal{G})$ and put $\mathcal{G}^{-1} = (g_1^{-1}, \ldots, g_m^{-1})$. We write F and $\langle\langle \Omega \rangle\rangle$ to denote the free group $F(x_1, \ldots, x_m)$ on $\{x_1, \ldots, x_m\}$ and the normal closure of Ω in F, respectively. In this subsection, we show that the first and second cohomology group of G are isomorphic to factor modules in T^m and T^Ω, respectively, where T^Ω denotes the set of sequences with entries in T and index set Ω. Let $\delta_\gamma : F \to T$ be defined as in the beginning of Section 4.2 and define

$$\begin{aligned} \tau_1 : \quad Z^1(G,T) \quad &\to T^m, \quad \psi \mapsto (\psi(g_1), \ldots, \psi(g_m)) \text{ and} \\ \tau_2 : \quad Z^2(G,T) \quad &\to T^\Omega, \quad \gamma \mapsto (\delta_\gamma(\omega) \mid \omega \in \Omega). \end{aligned}$$

Note that the homomorphisms τ_1 and τ_2 depend on the chosen polycyclic series $\mathcal{G}$. To shorten the notation, denote $Z_i := \tau_i(Z^i(G,T))$ and $B_i := \tau_i(B^i(G,T))$ for $i \in \{1,2\}$. In order to show that τ_1 is injective, we need the following elementary lemma.

Lemma 24 ([12, Proposition 2.73]). *Let $\psi \in Z^1(G,T)$ be a 1-cocycle and $g = h_1 \cdots h_l$ an element of G with $l \in \mathbb{N}$ and $h_i \in \mathcal{G} \cup \mathcal{G}^{-1}$. Further, for $1 \le i \le l$ let v_i denote the product $h_i \cdots h_l$ and put $v_{l+1} := 1$. Then we have*

$$\psi(g) = \sum_{\substack{1 \le i \le l, \\ h_i \in \mathcal{G}}} \psi(h_i).v_{i+1} - \sum_{\substack{1 \le i \le l, \\ h_i \in \mathcal{G}^{-1}}} \psi(h_i^{-1}).v_i.$$

The following Lemma is a fundamental result for the computer calculation of the first and second cohomology group of polycyclic groups, see for instance [6].

Lemma 25.

(i) τ_1 *is injective, i.e.* $Z_1 \cong Z^1(G,T)$, $B_1 \cong B^1(G,T)$ *and* $Z_1/B_1 \cong H^1(G,T)$*;*

(ii) $\operatorname{Ker}\tau_2$ *is contained in* $B^2(G,T)$*, i.e.* $Z_2/B_2 \cong H^2(G,T)$*.*

Proof. (i) The result for (i) follows by Lemma 24.
(ii) Let $\gamma \in \operatorname{Ker}\tau_2$ and let $\varphi : \langle x_1,\ldots,x_m \mid \Omega\rangle \to E(\gamma)$ be the group homomorphism defined by the images $\varphi(x_i\langle\langle\Omega\rangle\rangle) := (g_i, 0)$. Since $\tau_2(\gamma)$ is trivial, the group homomorphism φ is well-defined. Then the image of φ is a complement to T in $E(\gamma)$ and thus $\gamma \in B^2(G,T)$. $\square$

Recall that $E(0)$ denotes the group extension defined by $0 \in Z^2(G,T)$; in particular $E(0)$ is a split extension. For $s = (s_1,\ldots,s_m) \in T^m$ let $\tau_s : F \to E(0)$ be the homomorphism defined by $\tau_s(x_i) = (g_i, s_i)$ for $1 \le i \le m$. Write δ_s to denote the map

$$\delta_s : F \to T,\ \omega \mapsto \delta_s(\omega) \text{ with } \tau_s(\omega) = (\omega(\mathcal{G}), \delta_s(\omega)),$$

and let σ be the following homomorphism of abelian groups

$$\sigma : T^m \to T^\Omega,\ s \mapsto (\delta_s(\omega) \mid \omega \in \Omega).$$

Lemma 26. *The kernel and the image of* σ *are* Z_1 *and* B_2*, respectively.*

A proof of Lemma 26 is given in [6], where Eick introduced the map σ by giving the explicit images of the relators such as in Lemma 27. We give an alternative proof.

Proof. First, we show that the kernel of σ equals Z_1. Let $\psi \in C^1(G,T)$ be a map. Then ψ is a 1-cocycle if and only if $\{(g,\psi(g)) \mid g \in G\}$ is a subgroup of $E(0)$. This is the case if and only if the map $\delta_s : F \to T$ with $s = (\psi(g_1),\ldots,\psi(g_m))$ extends to a group homomorphism $\delta_s' : \langle x_1,\ldots,x_m \mid \Omega\rangle \to T,\ \omega\cdot\langle\langle\Omega\rangle\rangle \mapsto \delta_s(\omega)$. Thus $\psi \in Z^1(G,T)$ is equivalent to $\sigma(s) = 0$. It follows that $\operatorname{Ker}\sigma = \tau_1(Z^1(G,T)) = Z_1$.
It remains to show that the image of σ is B_2. For this purpose, let τ_1^* and $\alpha_\bullet$ be the homomorphisms

$$\begin{array}{lll} \tau_1^* : & C^1(G,T) \to T^m, & \psi \mapsto (\psi(g_1),\ldots,\psi(g_m)) \text{ and} \\ \alpha_\bullet : & C^1(G,T) \to Z^2(G,T), & \psi \mapsto \alpha_\psi, \end{array}$$

where α_ψ denotes the map $G\times G \to T$, $(g,h) \mapsto -\psi(gh)+\psi(g).h+\psi(h)$. Note that τ_1^* is surjective and the image of $\alpha_\bullet$ is $B^2(G,T)$. It follows that $B_2 = \operatorname{Im}(\tau_2 \circ \alpha_\bullet)$ and $\operatorname{Im}\sigma = \operatorname{Im}(\sigma \circ \tau_1^*)$. Thus it suffices to verify that the following diagram commutes in order to show the equality $\operatorname{Im}\sigma = B_2$.

$$\begin{tikzcd} C^1(G,T) \arrow[r, two heads, "\tau_1^*"] \arrow[d, "\alpha_\bullet"] & T^m \arrow[d, "\sigma"] \\ Z^2(G,T) \arrow[r, "\tau_2"] & T^\Omega \end{tikzcd}$$

Let $\psi \in C^1(G,T)$. Since G is finite, the subgroup $V := \langle \psi(g) \mid g \in G\rangle$ is finitely generated of rank d, say. Let $\{t_1,\ldots,t_d\}$ be a generating set of V and let $\gamma \in Z^2(G,V)$ be the cocycle induced by α_ψ, that is, $\gamma(g,h) = \alpha_\psi(g,h)$ for g, $h \in G$. Further, let

$$\langle x_1,\ldots,x_m,y_1,\ldots,y_d \mid \Omega(\gamma)\rangle := \operatorname{pc}(\gamma)$$

be defined as in Remark 22, in particular $\mathrm{pc}(\gamma)$ is a group presentation of $E(\gamma)$. Put $N := \langle\langle \Omega(\gamma) \rangle\rangle$ and let $\theta : \mathrm{pc}(\gamma) \to E(\gamma)$ denote the homomorphism defined by $\theta(x_i \cdot N) := (g_i, 0)$ and $\theta(y_j \cdot N) := (1, t_j)$. We write θ_1 to denote the map $E(0) \to E(\gamma)$, $(g, t) \mapsto (g, -\psi(g) + t)$. By extension theory, the map θ_1 is a group isomorphism. Let θ_2 be the composition $\theta_1^{-1} \circ \theta$

$$\begin{tikzcd}
E(\gamma) & \mathrm{pc}(\gamma) \arrow[l, "\theta"'] \arrow[dl, "\theta_2"] \\
E(0) \arrow[u, "\theta_1"] &
\end{tikzcd}$$

and denote $s := (\psi(g_1), \ldots, \psi(g_m))$. Note that the images $\theta_2(x_i)$ and $\theta_2(y_j)$ equal $(g_i, \psi(g_i))$ and $(1, t_j)$, respectively, and θ_2 is a group homomorphism. This yields for $\omega \in \Omega = \Omega(\mathcal{G}) \leq F(x_1, \ldots, x_m)$ and $\omega' = \prod_{i=1}^{d} y_i^{a_i} \in F(y_1, \ldots, y_d)$ that $\theta_2(\omega \cdot \omega' \cdot N) = (1, \delta_s(\omega) + \sum_{i=1}^{d} a_i \cdot t_i)$. Let ω be an element of Ω and recall that there exists a unique word $\omega_\gamma \in F(Y)$ with $\omega \cdot \omega_\gamma \in \Omega(\gamma)$. By Remark 22 we have $\theta_2(\omega_\gamma \cdot N) = (1, -\delta_\gamma(\omega))$ and consequently $\theta_2(\omega \cdot \omega_\gamma \cdot N) = (1, \delta_s(\omega) - \delta_\gamma(\omega))$. As θ_2 is a group homomorphism, the image $\theta_2(\omega \cdot \omega_\gamma \cdot N) = \theta_2(1 \cdot N)$ is trivial and hence $\delta_s(\omega) = \delta_\gamma(\omega)$. It follows that $\tau_2(\alpha_\psi) = \sigma(\tau_1^*(\psi))$ and thus the result. $\square$

The next Lemma is quite helpful to determine a transformation matrix of σ.

Lemma 27. *Let $a_{i,j,k}$ be as in the equations (4.1.1) - (4.1.3) and put $a_{i,i,i} := r_i$ and $a_{i,j,i} := 1$, where $r_1, \ldots, r_m$ denote the relative orders of $\mathcal{G}$. For $1 \leq i \leq j \leq m$ and $k \geq i$ denote $\omega_{i,j}(k) := (\sum_{l=0}^{a_{i,j,k}-1} g_k^l) \cdot g_{k+1}^{a_{i,j,k+1}} \cdots g_m^{a_{i,j,m}} \in \mathbb{Z}_p G$. Then for $s = (s_1, \ldots, s_m) \in T^m$ and for natural numbers i, j with $1 \leq i \leq m$ and $i < j \leq m$ we have*

$$\begin{aligned}
\delta_s(\omega_{i,i}) &= \textstyle\sum_{k=i}^{m} s_k.\omega_{i,i}(k), \\
\delta_s(\omega_{i,j}) &= \textstyle\sum_{k=i+1}^{m} s_k.\omega_{i,j}(k) + s_i.(-1 + g_i^{-1} g_j g_i) + s_j.(g_j^{-1} g_i) \text{ and} \\
\delta_s(\omega_{j,i}) &= \textstyle\sum_{k=i+1}^{m} s_k.\omega_{j,i}(k) + s_i.(g_i^{-1} - g_j^{-1} g_i^{-1}) + s_j.(g_j^{-1} g_i^{-1}).
\end{aligned}$$

Proof. The equations are straightforward to observe. $\square$

If G is finite and T is torsion-free, then Z_2 can be deduced from B_2.

Lemma 28. *Assume that G is finite and T is torsion-free. Then Z_2/B_2 is the torsion subgroup of T^Ω/B_2.*

Proof. Let M/B_2 be the torsion subgroup of T^Ω/B_2. For $v \in M$ there exists $m \in \mathbb{N}$ with $m \cdot v \in B_2$. Since T is torsion-free, it follows that $v \in Z_2$ and thus $M \leq Z_2$. Conversely, let v be an element of Z_2. Then $|G| \cdot v \in B_2$ and hence $v \in M$. $\square$

Chapter 5

Coclass theory

Coclass theory has provided significant new insights into the theory of finite p-groups. In Section 5.1 we recall some basic definitions of coclass theory and some results about pro-p-groups of finite coclass. Then we define coclass families in Section 5.2. This is motivated by the results of [9]. The definition of coclass families is very similar to the one of infinite coclass sequences which can be found in [10]. Isomorphism classes of coclass families are considered in Section 5.3. In the first part of Section 5.3 we mainly follow [9]. As already mentioned there is a mistake in [9]. We fill the gap caused by the mistake in Section 5.3.1 which mainly consists of new results. Section 5.4 deals with the ultimate periodicity of shaved coclass trees. Following [9] we define graph isomorphisms between the branches of a shaved coclass tree. Eick & Leedham-Green have shown that the graph isomorphisms are well-defined. As a consequence of the mistake mentioned above their proof is incorrect and has to be rewritten in Section 5.4.

5.1 Basic definitions and pro-p-groups of finite coclass

For a p-group G we write $\mathrm{cl}(G)$ to denote the nilpotency class of G. Further, let

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \cdots$$

denote the lower central series of G, in particular $\gamma_{i+1}(G) = [G, \gamma_i(G)]$ for $i \in \mathbb{N}$. Recall that a *pro-p-group* is an inverse limit of finite p-groups.

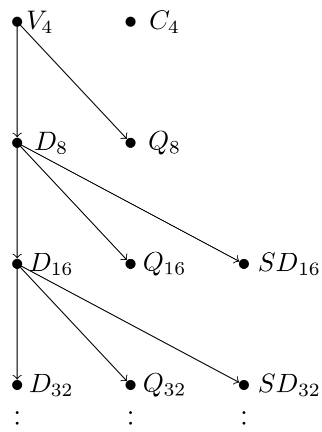
Definition 29. The *coclass* of

(i) a finite p-group G is defined as $\mathrm{cc}\, G := \log_p(|G|) - \mathrm{cl}(G)$;

(ii) an infinite pro-p-group G is defined as

$$\mathrm{cc}\, G := \begin{cases} \lim_{i\to\infty} \mathrm{cc}(G/\gamma_i(G)), & \text{if every quotient } G/\gamma_i(G) \text{ is finite,} \\ \infty, & \text{otherwise.} \end{cases}$$

For a prime p and a natural number r let $\mathcal{G}(p,r)$ be a directed graph, whose vertices represent the isomorphism classes of the finite p-groups of coclass r; further, there is a directed edge from G to H if there exists a normal subgroup N of H with $|N| = p$ and $H/N \cong G$. We call $\mathcal{G}(p,r)$ *coclass*



V_4 : Klein four-group
D_n : Dihedral group of order n
Q_n : Quaternion group of order n
D_n : Semi-dihedral group of order n

Figure 5.1: Coclass graph $\mathcal{G}(2,1)$

graph. In Figure 5.1 the coclass graph of the 2-groups of coclass 1 is pictured. In the visualizations of the coclass graphs the groups of the same order are on the same level and groups of larger order are drawn below smaller ones.
The proven Coclass Conjectures C & D are fundamental for the structure of $\mathcal{G}(p,r)$.

Theorem 30 ([20, Coclass Conjectures C & D]). *For fixed prime p and natural number r there are only finitely many isomorphism classes of infinite pro-p-groups of coclass r. These pro-p-groups are soluble.*

The infinite paths in $\mathcal{G}(p,r)$ are induced by pro-p-groups: let S be an infinite pro-p-group of coclass r. By Theorem 30 there exists a minimal natural number $m = m(S)$ such that $S/\gamma_m(S)$ has coclass r and there is only one infinite pro-p-group R of coclass r with $R/\gamma_m(R) \cong S/\gamma_m(S)$ up to isomorphism. Denote $S_i := S/\gamma_i(S)$. Since S has finite coclass r, the groups S_i are finite. Further, the nilpotency class of S_{i+1} is $i+1$ and its order is at least $p\cdot|S_i|$. This yields $\mathrm{cc}\, S_{i+1} \geq \mathrm{cc}\, S_i$ and thus $\mathrm{cc}\, S_i = r$ for $i \geq m$. Hence $(S_i \mid i \geq m)$ is a family consisting of finite p-groups of coclass r and this family is an infinite path in $\mathcal{G}(p,r)$.

Definition 31. For $i \geq m = m(S)$ the group S_i is called *mainline group* and the infinite path in $\mathcal{G}(p,r)$ induced by the mainline groups S_m, S_{m+1}, $S_{m+2}, \ldots$ is called *mainline*. The group S_m is the *root* of the mainline.

We define an equivalence relation on the set of infinite paths in $\mathcal{G}(p,r)$: two paths are equivalent if they differ only in finitely many vertices. Note that the set of mainlines forms a transversal of this equivalence relation. Since the equivalence classes of the infinite paths in $\mathcal{G}(p,r)$ correspond 1-1 to the isomorphism classes of infinite pro-p-groups of coclass r by [9, p. 2], there are only finitely many mainlines.
Let $\mathbb{Z}_p$ denote the p-adic integers and let the rank of a $\mathbb{Z}_p$-module be defined as the rank of its Frattini quotient. Note that the Frattini quotient of a $\mathbb{Z}_p$-module N is N/pN and hence elementary abelian. In what follows we consider pro-p-groups of finite coclass. For this purpose we need

Definition 32 (cf. [20, Definition 4.1.1])**.** Let G be a finite p-group, acting on a $\mathbb{Z}_p$-module N of finite rank d.

(i) The *lower G-central series of N* is defined as

$$N = N_0 \geq N_1 \geq N_2 \geq \cdots,$$

where $N_0 := N$ and $N_{i+1} := N_{i+1}(G) := [G, N_i] = \langle n.g - n \mid n \in N_i, g \in G\rangle$ for each $i \geq 0$.

(ii) G acts *uniserially* on N, if $[N_i : N_{i+1}] = p$ for every i with $N_i \neq 0$.

Let G and N be as in the previous definition. Then G acts on $N/p \cdot N \cong \mathbb{F}_p^d$. Let $\mathfrak{X} : G \to \mathrm{GL}_d(\mathbb{F}_p)$ be the homomorphism induced by the action homomorphism of G on N. Since G and N are a finite p-group and a finite, elementary-abelian p-group, respectively, we may assume that the image of G under $\mathfrak{X}$ is a unitriangular matrix group. This yields

Lemma 33. *Let G be a finite p-group acting uniserially on a $\mathbb{Z}_p$-module N of finite rank d with lower G-central series $N = N_0 \geq N_1 \geq N_2 \cdots$. Then we have $N_{i+d} = p \cdot N_i$ for $i \in \mathbb{N}_0$.*

It will turn out that infinite pro-p-groups of finite coclass are uniserial p-adic pre-space groups. We use the definitions of translation subgroup, point group and uniserial p-adic pre-space group as given in [9, p. 3]:

Definition 34. Let S be a group and T a normal subgroup of S. If $P := S/T$ is finite and T is a free $\mathbb{Z}_p$-module of finite rank d, say, then T is called a *translation subgroup* of S and the dimension of S is defined as d. If additionally P is a p-group acting uniserially on T, then S and P are called a *uniserial p-adic pre-space group* and a *point group* of S, respectively.

To ease the notation we write translation subgroups, which are free $\mathbb{Z}_p$-modules, additively. Let S be a uniserial p-adic pre-space group and let T be a translation subgroup of S. Then pT is also a translation subgroup of S and thus S has infinitely many translation subgroups. However the dimension of S is well-defined: let U be another translation subgroup of S. Since S/T and S/U are finite, there exists $m \in \mathbb{N}$ with $mU \leq T$ and $mT \leq U$. It follows that $\operatorname{rank} U \leq \operatorname{rank} T$ and $\operatorname{rank} T \leq \operatorname{rank} U$, and hence the equality $\operatorname{rank} U = \operatorname{rank} T$. If S is also uniserial then S has finite coclass: let

$$\begin{array}{lccccccc} S = & \gamma_1(S) & \geq & \gamma_2(S) & \geq & \gamma_3(S) & \geq & \cdots \quad \text{and} \\ T = & T_0 & \geq & T_1 & \geq & T_2 & \geq & \cdots \end{array}$$

be the lower central series of S and the lower S-central series of T, respectively.

Lemma 35. *Let m be the nilpotency class of the point group $P = S/T$. Then there exists a nonnegative integer $i \leq m-1$ with $\gamma_m(S) = T_i$, in particular $\operatorname{cc} S = \operatorname{cc} P + i$.*

Proof. Evidently, the group T_{m-1} lies in $\gamma_m(S)$ and hence $\gamma_m(S) \neq 1$. It follows that there exists a nonnegative integer $i \leq m-1$ with $\gamma_m(S) \leq T_i$ and $\gamma_m(S) \not\leq T_{i+1}$. Since S acts uniserially on T and $\gamma_m(S)$ is a normal subgroup of S, we have $\gamma_m(S) = T_i$. The uniserial action of S/T on T yields $\gamma_{m+j}(S) = T_{i+j}$ and $|\gamma_m(S)/\gamma_{m+j}(S)| = p^j$ for $j \in \mathbb{N}_0$. It follows that $\operatorname{cc}(S/\gamma_{m+j}(S)) = \operatorname{cc}(S/T_i) + j - j$ and hence $\operatorname{cc} S = \operatorname{cc}(S/T_i)$. As S/T and S/T_i have the same nilpotency class, we have $\operatorname{cc} S = \operatorname{cc}(S/T_i) = \operatorname{cc}(S/T) + i$. □

By Lemma 35 every infinite pro-p-group which is a uniserial p-adic pre-space group has finite coclass. The reverse also holds.

Lemma 36 ([20, Corollary 7.4.13])**.** *Let S be an infinite pro-p-group of finite coclass r, say. Then S is a uniserial p-adic pre-space group of dimension $p^s(p-1)$ for some s with $0 \leq s \leq r-1$ for p odd, and $0 \leq s \leq r+1$ for $p=2$.*

5.2 Coclass families

In this section we define coclass families. Throughout this section we write P and T to denote a finite p-group and a $\mathbb{Z}_p P$-module, respectively. Let T_i denote the $(i+1)$-th term of the lower P-central series of T. Further for a 2-cochain $\delta \in C^2(P,T)$ and $n \in \mathbb{N}$ we write $\delta_{T/T_n} \in C^2(P, T/T_n)$ to denote the composition of δ and the natural surjection $T \twoheadrightarrow T/T_n$, which is a P-module homomorphism. Recall that for a 2-cocycle γ we write $E(\gamma)$ to denote the group extension defined by γ. Motivated by the results of [9] we introduce coclass families.

Definition 37. Let P, T be groups and let $\gamma \in Z^2(P,T)$, $\delta \in C^2(P,T)$ and $n \in \mathbb{N}$ be such that

(i) P is a finite p-group,

(ii) T is a free $\mathbb{Z}_p$-module of finite rank $d \in \mathbb{N}$ on which P acts uniserially,

(iii) the coclass of the group extension $E(\gamma)$ is equal to $\operatorname{cc} P$ and

(iv) $\delta_{T/T_n} \in Z^2(P, T/T_n)$.

Then the family $\mathcal{F} = \mathcal{F}(\gamma,\delta,n) := (G_k \mid k \geq 0)$ defined by $G_k := E(\gamma_{T/T_{n+kd}} + p^k \delta_{T/T_{n+kd}})$ is called a *coclass family*. Further, $S := E(\gamma)$, P, T and d are called the associated pro-p-group, point group, translation subgroup and dimension of the coclass family $\mathcal{F}$.

Remark 38.

(i) *Every pro-p-group S of finite coclass induces coclass families: Let T be a translation subgroup of S. Then by Lemma 35 we have $\operatorname{cc} S/T_i = \operatorname{cc} S$ for some $i \in \mathbb{N}_0$. Hence we may choose T such that $\operatorname{cc} S = \operatorname{cc} S/T$. Put $P := S/T$ and let $\gamma \in Z^2(P,T)$ be with $E(\gamma) \cong S$. Then every $n \in \mathbb{N}$ and $\delta \in C^2(P,T)$ satisfying $\delta_{T/T_n} \in Z^2(P,T/T_n)$ defines a coclass family $\mathcal{F}(\gamma,\delta,n)$.*

(ii) *Let $\mathcal{F}(\gamma,\delta,n)$ be a coclass family and let $S = E(\gamma)$, P, T and d be the associated pro-p-group, point group, translation subgroup and dimension of $\mathcal{F}(\gamma,\delta,n)$. Then S is a uniserial-p-adic pre-space group of dimension d and coclass $\operatorname{cc} P$. The group $\tilde{T} := \{(1,t) \mid t \in T\} \cong T$ is a translation subgroup of S with corresponding point group $S/\tilde{T} \cong P$.*

Coclass families are closely related to infinite coclass sequences introduced in [10]: every infinite coclass sequence is a coclass family, however the reverse is not true. But for every coclass family $(G_k \mid k \geq 0)$ there exists a natural m such that $(G_k \mid k \geq m)$ is an infinite coclass sequence.
In what follows, let $\mathcal{F}(\gamma,\delta,n) = (G_k \mid k \geq 0)$ denote a coclass family with associated pro-p-group $S = E(\gamma)$, point group P and translation subgroup T of rank $d \in \mathbb{N}$. All but finitely many groups of a coclass family have the same coclass:

Lemma 39. *For $k \geq 1$ the coclass of G_k is $\operatorname{cc} P$. If δ_{T/T_d} is trivial, then we also have $\operatorname{cc} G_0 = \operatorname{cc} P$.*

Proof. By Remark 38 the group extension $S = E(\gamma)$ is a uniserial p-adic pre-space group of coclass $\operatorname{cc} P$ with translation subgroup $\tilde{T} := \{(1,t) \mid t \in T\} \cong T$ and corresponding point group $E(\gamma)/\tilde{T} \cong P$. Let m denote the nilpotency class of P. Then Lemma 35 yields $\gamma_m(E(\gamma)) = \{(1,t) \mid t \in T\}$. For $k \geq 1$ we have $(\gamma + p^k\delta)_{T/T_d} = \gamma_{T/T_d}$ and hence $\gamma_m(G_k) = \{(1, t+T_{n+kd}) \mid t \in T\}$. The result follows by the uniserial action of P on T. $\square$

By the uniserial action of P on T we have $T_{kd} = p^k T$ for $k \in \mathbb{N}_0$. Obviously, it is easier to cope with factor modules of the form $T/p^kT \cong (\mathbb{Z}/p^k\mathbb{Z})^d$ than with T/T_n for arbitrary n.

Lemma 40. *There exists a multiple m of the rank d of T and a coclass family $\mathcal{F}(\beta, \epsilon, m) = (H_k \mid k \geq 0)$ such that $H_k \cong G_{k+1}$ for every $k \geq 0$.*

Proof. Recall that G_k is defined as $E(\gamma_{T/T_{n+kd}} + p^k \delta_{T/T_{n+kd}})$. Let $0 \leq l < d$ be the nonnegative integer satisfying $l = n \mod d$, where n arises from $\mathcal{F}(\gamma, \delta, n) = (G_k \mid k \geq 0)$. Put $Q := E(\gamma_{T/T_l})$. Evidently, there exists a 2-cocycle $\beta \in Z^2(Q, T_l)$ such that $E(\gamma)$ is naturally isomorphic to $E(\beta)$. Since $p \cdot \delta_{T/T_l} = 0$, there exists also $\epsilon \in C^2(Q, T_l)$ with $E(\beta_{T_l/T_{n+(k+1)d}} + p^k \epsilon_{T_l/T_{n+(k+1)d}}) \cong E(\gamma_{T/T_{n+(k+1)d}} + p^{k+1}\delta_{T/T_{n+(k+1)d}}) = G_{k+1}$ for every $k \geq 0$. As $n - l$ is a multiple of d, the result follows. □

5.3 Isomorphism classes of coclass families

We say that two coclass families $(G_k \mid k \geq 0)$ and $(H_k \mid k \geq 0)$ are isomorphic to each other if there exists an integer i with $G_k \cong H_{k+i}$ for all but finitely many $k \geq \min\{0, -i\}$.

Let P be a finite p-group acting uniserially on a free $\mathbb{Z}_p$-module T of finite rank $d \in \mathbb{N}$. Further, let $T = T_0 \geq T_1 \geq T_2 \geq \cdots$ be the lower P-central series of T and let n denote a nonnegative integer. Recall that for $k \in \mathbb{N}_0$ the group T_{n+kd} equals $p^k T_n$ by the uniserial action of P on T. In the remaining part of this section, we consider the cohomology groups $H^2(P, T/T_{n+kd})$ and the group of compatible pairs $\mathrm{Comp}(P, T/T_{n+kd})$. We shall see that there are only finitely many coclass families with fixed associated point group P and translation subgroup T up to isomorphism. In order to apply results from Chapter 2 on $H^2(P, T/T_n)$ and $\mathrm{Comp}(P, T/T_n)$ it is necessary that n is large enough. In what follows, we give a sufficiently large lower bound for n.

Let $t_0 \in T$ be such that $\langle t_0, T_1 \rangle = T$ and let P_{t_0} denote the stabilizer of t_0 in P. Put $a := a(n) := \max\{\exp H^2(P,T), \exp H^3(P, T_n)\}$ and $b := b(n) := \exp H^1(P_{t_0}, T_n)$. Since T_m is torsion-free and $T_{m+d} = p \cdot T_m$ for $m \in \mathbb{N}_0$, the groups $H^3(P, T_m)$ and $H^1(P_{t_0}, T_m)$ are naturally isomorphic to $H^3(P, T_n)$ and $H^1(P_{t_0}, T_n)$ for $m = n \mod d$, respectively. Hence we find a natural number n such that

$$p^{n/d} \geq \max\{a, b\} \cdot b. \tag{5.3.1}$$

It will turn out in Lemma 49 that b does not depend on the choice of t_0. Let us recall a result from Chapter 2: For every $k \in \mathbb{N}_0$ the embedding $T_{n+kd} \hookrightarrow T$ and the natural homomorphism $\vartheta : T \twoheadrightarrow T/T_{n+kd}$ induces an exact sequence

$$Z^2(P,T) \xrightarrow{\vartheta_{2,k}} Z^2(P, T/T_{n+kd}) \xrightarrow{\delta_{2,k}} H^3(P, T_{n+kd}), \tag{5.3.2}$$

where $\vartheta_{2,k}$ is defined as $Z^2(P,T) \to Z^2(P, T/T_{n+kd})$, $\gamma \mapsto \vartheta \circ \gamma$ and $\delta_{2,k}$ is the connecting homomorphism from the Snake Lemma, see Lemma 7 and in particular Exact Sequence (2.2.2). Assumption (5.3.1) yields that T_n is a submodule of $\max\{\exp H^2(P,T), \exp H^3(P,T_n)\} \cdot T$. Hence by Lemma 9 the Sequence (5.3.2) gives rise to a split short exact sequence $0 \to \mathrm{Im}\,\vartheta_{2,k} \hookrightarrow Z^2(P, T/T_{n+kd}) \to H^3(P, T_{n+kd}) \to 0$, inducing the following split short exact sequence of cohomology groups

$$0 \to H^2(P,T) \to H^2(P, T/T_{n+kd}) \to H^3(P, T_{n+kd}) \to 0. \tag{5.3.3}$$

This enables us to define a group isomorphism μ_k from $H^2(P, T/T_n)$ to $H^2(P, T/T_{n+kd})$. Here we follow mainly [9]. Let $\delta'_{2,0} : H^3(P, T_n) \to Z^2(P, T/T_n)$ be a right inverse of the connecting homomorphism $\delta_{2,0} : Z^2(P, T/T_n) \to H^3(P, T_n)$ in Sequence (5.3.2). Further, let $\delta'_{2,k}$ be the unique group homomorphism such that the following diagram commutes

$$\begin{tikzcd} Z^2(P,T/T_n) \arrow[d] & H^3(P,T_n) \arrow[l, "\delta'_{2,0}"'] \arrow[d] \\ Z^2(P,T/T_{n+kd}) & H^3(P,T_{n+kd}) \arrow[l, "\delta'_{2,k}"'] \end{tikzcd}$$

where the vertical arrows are induced by the P-module homomorphism $T \to T$, $t \mapsto p^k t$. Then there exists a unique group isomorphism $\mu_k = \mu(n)_k$ such that the following diagram is commutative

$$\begin{tikzcd} H^2(P,T) \arrow[r] \arrow[d, equal, "\mathrm{id}"'] & H^2(P,T/T_n) \arrow[d, "\mu_k"'] & H^3(P,T_n) \arrow[l] \arrow[d] \\ H^2(P,T) \arrow[r] & H^2(P,T/T_{n+kd}) & H^3(P,T_{n+kd}) \arrow[l] \end{tikzcd}$$

where the group isomorphism $H^3(P, T_n) \to H^3(P, T_{n+kd})$ is induced by $t \mapsto p^k t$ and the vertical arrows are induced by $\vartheta_{2,0}$, $\delta'_{2,0}$ and $\vartheta_{2,k}$, $\delta'_{2,k}$.
Note that a is at least $\exp H^3(P, T_n) = \exp \operatorname{Im} \delta'_{2,0}$. The image of an element $\tau + B^2(P, T/T_n)$ under μ_k can be explicitly described in the following way: let $\gamma \in Z^2(P, T)$ and $\delta \in C^2(P, T_{n-\log_p(a)d})$ be such that the sum $\gamma_{T/T_n} + \delta_{T/T_n}$ equals τ and δ_{T/T_n} lies in $\operatorname{Im} \delta'_{2,0}$. Then we have

$$\mu_k(\tau + B^2(P, T/T_n)) \quad = \gamma_{T/T_{n+kd}} + p^k \delta_{T/T_{n+kd}} + B^2(P, T/T_{n+kd}). \tag{5.3.4}$$

The group isomorphism μ_k has motivated the definition of coclass families.

Lemma 41. *Let $\mathcal{F}(\gamma, \delta, n)$ be a coclass family with associated point group P and translation subgroup T. Assume that n satisfies Inequality (5.3.1). Then there exists an element $\epsilon \in \operatorname{Im} \delta'_{2,0} \leq Z^2(P, T_{n-\log_p(a)d}/T_n)$ such that for k with $p^k \geq \exp H^2(P, T)$ the 2-cocycle $(\gamma + p^k \delta)_{T/T_{n+kd}}$ lies in the coset of $\mu_k(\gamma_{T/T_n} + \epsilon + B^2(P, T/T_n))$.*

Proof. Evidently, there exists $\epsilon \in \operatorname{Im} \delta'_{2,0}$ with $\delta_{2,0}(\epsilon) = \delta_{2,0}(\delta)$. Thus $p^k(\epsilon - \delta)_{T/T_{n+kd}}$ is a 2-coboundary for k with $p^k \geq \exp H^2(P, T)$. The result follows. □

Lemma 41 states that every coclass family with associated point group P and translation subgroup T is isomorphic to a coclass family arising from an element of $H^2(P, T/T_n)$ and its images under μ_k. Since $H^2(P, T/T_n)$ is finite, we obtain

Corollary 42. *Up to isomorphism, there are only finitely many coclass families with fixed associated point group and translation subgroup.*

We have seen in Corollary 42 that there are only finitely many isomorphism classes of coclass families corresponding to a fixed point group and translation subgroup. However this does not give us any criteria to decide whether two coclass families are isomorphic, or not. In this context the following natural questions arise: do finitely many groups in a coclass family determine the isomorphism class of the family? Could it be that there are two coclass families $(G_k \mid k \geq 0)$ and $(H_k \mid k \geq 0)$ which are not isomorphic to each other but $G_k \cong H_l$ hold for infinitely many k and l? An answer to this questions is given by

Theorem 43. *Let $\mathcal{F} = \mathcal{F}(\gamma, \delta, n)$ and $\mathcal{F}' = \mathcal{F}(\gamma', \delta', n)$ be coclass families with associated point group P and translation subgroup T. Assume that n satisfies Inequality (5.3.1) and that δ_{T/T_n}, $\delta'_{T/T_n} \in \operatorname{Im} \delta'_{2,0} \leq Z^2(P, T_{n-\log_p(a)d}/T_n)$. Then the coclass family $\mathcal{F} = (G_k \mid k \geq 0)$ is isomorphic to $\mathcal{F}' = (H_k \mid k \geq 0)$ if and only if $G_0 \cong H_0$. Further, $\mathcal{F} \cong \mathcal{F}'$ implies $G_k \cong H_k$ for every $k \geq 0$.*

Note that the assumption δ_{T/T_n}, $\delta'_{T/T_n} \in \operatorname{Im} \delta'_{2,0}$ in Theorem 43 is not really a restriction by Lemma 41.

5.3.1 Proof of Theorem 43

In this subsection we assume that n satisfies Inequality (5.3.1), and we write A_k to denote the factor P-module T/T_{n+kd}. Further, let $\mu_k : H^2(P, A_0) \to H^2(P, A_k)$ be the group isomorphism introduced in the previous section. For a P-module V let $\operatorname{Comp}(P, V)$ denote the group of compatible pairs of P and V, that is,

$$\operatorname{Comp}(P, V) = \{(\beta, \epsilon) \in \operatorname{Aut} P \times \operatorname{Aut} V \mid \overline{g^{\beta}} = \overline{g}^{\epsilon} \text{ for all } g \in P\},$$

where $\bar{\cdot} : P \to \operatorname{Aut} V$, $g \mapsto \overline{g}$ denotes the group homomorphism induced by the group action of P on V. In order to prove Theorem 43 it is essential to know

Theorem 44 ([9, Theorem 20])**.** *Let V be a P-module. Let δ, $\beta \in H^2(P, V)$ be such that their corresponding extensions $E(\delta)$ and $E(\beta)$ of V by P have coclass* $\operatorname{cc} P$*. Then $E(\delta) \cong E(\beta)$ if and only if δ and β lie in the same orbit under the action of* $\operatorname{Comp}(P, V)$*.*

Theorem 44, Lemma 39 and Equation (5.3.4) yield that Theorem 43 is equivalent to

Theorem 45. *For $k \geq 0$, the homomorphism $\mu_k : H^2(P, A_0) \to H^2(P, A_k)$ induces a* 1-1 *correspondence between the orbits of $H^2(P, A_0)$ and $H^2(P, A_k)$ under the action of* $\operatorname{Comp}(P, A_0)$ *and* $\operatorname{Comp}(P, A_k)$*, respectively.*

The original proof of Theorem 45 given by Eick & Leedham-Green in [9] is based on [9, Theorem 23]. It turned out that [9, Theorem 23] is false, see Appendix B. This incorrect theorem deals with compatible pairs. Hence in order to prove Theorem 45 we need deeper insight into the structure of $\operatorname{Comp}(P, A_k)$. For this purpose, we consider the P-endomorphisms $A_k \to A_k$ in the following paragraph. Note that $(1, \beta) \in \operatorname{Aut} P \times \operatorname{Aut} A_k$ is a compatible pair if and only if β is a P-endomorphism.

P-Homomorphisms

For $\mathbb{Z}_p P$-modules V and W we write $\operatorname{Hom}_P(V, W)$ to denote the group of $\mathbb{Z}_p P$-module homomorphisms $V \to W$, and we define $\operatorname{End}_P V := \operatorname{Hom}_P(V, V)$. Since P acts on $T \cong \mathbb{Z}_p^d$, the group T can be considered as $\mathbb{Z}_p P$-module.

Lemma 46. *The module $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is simple as $\mathbb{Q}_p P$-module.*

Proof. Let U be a submodule of $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then there exists $i \in \mathbb{N}_0$ with $U \leq T_i \otimes \mathbb{Q}_p$ and $U \not\leq T_{i+1} \otimes \mathbb{Q}_p$. By the uniserial action of P on T it follows that $T_j \otimes \mathbb{Q}_p \leq U$ for $j \geq i$. This yields $T_{kd} \otimes \mathbb{Q}_p \leq U$ for some $k \in \mathbb{N}_0$. Since $T_{kd} = p^k T$, we have $p^k T \otimes \mathbb{Q}_p = T \otimes \mathbb{Q}_p \leq U$. $\square$

Schur's Lemma yields

Lemma 47. $\mathrm{End}_P(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is a skew field.*

Let V and W be $\mathbb{Z}_pP$-modules and assume that P acts uniserially on V, for instance $V = T$ or $V = A_k$. As usual, we write $V = V_0 \geq V_1 \geq V_2 \geq \cdots$ to denote the lower P-central series of V. For an element $v \in V$ and a subgroup H of P let P_v and $C_V(H)$ denote the stabilizer of v in P and the $\mathbb{Z}_p$-module of elements of V fixed by H, respectively, that is,

$$P_v := \mathrm{Stab}_P(v) := \{g \in P \mid v.g = v\} \text{ and } C_V(H) := \{v \in V \mid v.g = v \text{ for } g \in H\}.$$

Lemma 48. *Let $v_0 \in V$ be such that $\langle v_0, V_1\rangle = V$ and define $\eta : \mathrm{Hom}_P(V, W) \to W$, $\varphi \mapsto \varphi(v_0)$. Then η is a $\mathbb{Z}_p$-module monomorphism and* $\mathrm{Im}\,\eta = C_W(P_{v_0})$.

Proof. Put $f := \mathrm{rank}\, V$ and let $v_1, \ldots, v_{f-1}$ be elements of V satisfying $v_i \in [P, \langle v_0, \ldots, v_{i-1}\rangle]$ and $\langle v_i, V_{i+1}\rangle = V_i$ for $1 \leq i \leq f-1$. Note that the image of v_i under a P-homomorphism φ is uniquely determined by $\varphi(v_0)$. Since P acts uniserially on V, we have $\bigoplus_{i=0}^{f-1} \mathbb{Z}_p v_i = V$. Hence η is a $\mathbb{Z}_p$-module monomorphism. It remains to show that $\mathrm{Im}\,\eta = C_W(P_{v_0})$. Let $\varphi \in \mathrm{Hom}_P(V, W)$. Then for every $g \in P_{v_0}$ we have $\varphi(v_0).g = \varphi(v_0.g) = \varphi(v_0)$ and thus the element $\varphi(v_0) = \eta(\varphi)$ lies in $C_W(P_{v_0})$. Conversely, let w be an element of $C_W(P_{v_0})$ and define $\varphi : V \to W$ via $\varphi(v_0) = w$ and $\varphi(v_0.\sum_{g\in P} a_g g) := \sum_{g\in P} a_g w.g$ for $\sum_{g\in P} a_g g \in \mathbb{Z}_pP$. We show that φ is well-defined. Let $g, g' \in P$ with $v_0.g = v_0.g'$, in particular $g'g^{-1} \in P_{v_0}$. Since $w \in C_W(P_{v_0})$, we have $w.g'g^{-1} = w$ and thus $\varphi(v_0.g') = w.g' = (w.g'g^{-1}).g = w.g = \varphi(v_0.g)$. The result follows. $\square$

Compatible pairs

Let us now consider the structure of $\mathrm{Comp}(P, A_k)$. To ease the notation put

$$\Gamma := \mathrm{Comp}(P, T) \text{ and } \Gamma_k := \mathrm{Comp}(P, A_k).$$

For an automorphism $\beta \in \mathrm{Aut}\, P$ and a nonnegative integer k let $T_k^{(\beta)}$ denote the G-module, which is equal to T_k as $\mathbb{Z}_p$-module and on which G acts via $t * g := t.(g^\beta)$. Further denote $A_k^{(\beta)} := T^{(\beta)}/T^{(\beta)}_{n+kd}$, $\mathrm{End}_P^\beta\, T := \mathrm{Hom}_P(T, T^{(\beta)})$ and $\mathrm{End}_P^\beta\, A_k := \mathrm{Hom}_P(A_k, A_k^{(\beta)})$. Note that

$$\Gamma = \{(\beta, \epsilon) \in \mathrm{Aut}\, P \times \mathrm{Aut}\, T \mid \epsilon \in \mathrm{End}_P^\beta\, T\} \text{ and} \tag{5.3.5}$$

$$\Gamma_k = \{(\beta, \epsilon) \in \mathrm{Aut}\, P \times \mathrm{Aut}\, A_k \mid \epsilon \in \mathrm{End}_P^\beta\, A_k\}. \tag{5.3.6}$$

For a homomorphism $\epsilon : T \to T$ and a nonnegative integer l with $\epsilon(T_{n+ld}) \subseteq T_{n+ld}$ let $\epsilon_{A_l} : A_l \to A_l$ be the homomorphism induced by $T \twoheadrightarrow A_l$. Similar we define $\epsilon_{A_l} : A_l \to A_l$ for $\epsilon : A_k \to A_k$ with $k \geq l$ and $\epsilon(T_{n+ld}/T_{n+kd}) \subseteq T_{n+ld}/T_{n+kd}$. Recall that t_0 denotes an element with $\langle t_0, T_1\rangle = T$ and that $p^{n/d}$ is at least $\exp H^1(P_{t_0}, T)$ by Assumption (5.3.1).

Lemma 49. *For every $\beta \in \mathrm{Aut}\, P$ there exists a group E_β of maps $T \to T$ such that for every nonnegative integer k we have*

$$\mathrm{End}_P^\beta\, A_k = (\mathrm{End}_P^\beta\, T)_{A_k} \oplus (p^k E_\beta)_{A_k}$$

and $(p^k E_\beta)_{A_k}$ is isomorphic to $H^1(P_{t_0}, T_{n+kd})$.

Proof. Let ϑ denote the natural homomorphism $T \twoheadrightarrow A_0$ and put $Q := P_{t_0}$. First, we show that $H^1(Q, T_n^{(\beta)}) \cong H^1(Q, T_n)$ and $H^0(Q, A_0^{(\beta)}) \cong \vartheta(H^0(Q, T^{(\beta)})) \oplus H^1(Q, T_n^{(\beta)})$. Let $\alpha : Z^1(Q, T_n^{(\beta)}) \to Z^1(Q, T_n)$ be defined by $\alpha(\psi)(g) := \psi(g^\beta)$ for $\psi \in Z^1(P, T_n^\beta)$ and $g \in Q$. It is straightforward to check that α is a well-defined group isomorphism and that the image of $B^1(P, T_n^{(\beta)})$ is $B^1(P, T_n)$. It follows that $H^1(Q, T_n)$ is isomorphic to $H^1(Q, T_n^{(\beta)})$, in particular $\exp H^1(Q, T_n)$ equals $\exp H^1(Q, T_n^{(\beta)})$. Thus we have $p^{n/d} \geq \exp H^1(Q, T_n^{(\beta)})$ by Assumption (5.3.1). In particular $T_n^{(\beta)}$ is a P-submodule of $\exp H^1(Q, T_n^{(\beta)}) \cdot T^{(\beta)}$ by the uniserial action of P on $T^{(\beta)}$. Then Lemma 9 yields $H^0(Q, A_0^{(\beta)}) \cong \vartheta(H^0(Q, T^{(\beta)})) \oplus H^1(Q, T_n^{(\beta)})$.

Next we show that $H^0(P_{t_0+T_n}, A_0^{(\beta)})$ equals $H^0(Q, A_0^{(\beta)})$. For $k \in \mathbb{N}_0$ the P-module isomorphism $T^{(\beta)} \to T_{kd}^{(\beta)}$, $t \mapsto p^k t$ induces a natural embedding of the group $\operatorname{End}_P^\beta A_0$ into $\operatorname{End}_P^\beta A_k$, whose image is $\operatorname{Hom}_P(A_k^{(\beta)}, p^k \cdot A_k^{(\beta)})$. Since P is finite, the group $P_{t_0+T_{n+kd}}$ equals P_{t_0} for some k. By Lemma 48 we have $\operatorname{End}_P^\beta A_0 \cong H^0(P_{t_0+T_n}, A_0^{(\beta)})$ and $\operatorname{End}_P^\beta A_k \cong H^0(P_{t_0}, A_k^{(\beta)})$. Then the embedding $\operatorname{End}_P^\beta A_0 \hookrightarrow \operatorname{End}_P^\beta A_k$ yields $H^0(Q, A_0^{(\beta)}) \cong H^0(P_{t_0}, p^k \cdot A_k^{(\beta)})$. As $H^0(P_{t_0}, p^k \cdot A_k^{(\beta)})$ is canonically isomorphic to $H^0(P_{t_0}, A_0^{(\beta)})$, it follows that $H^0(P_{t_0+T_n}, A_0^{(\beta)}) = H^0(Q, A_0^{(\beta)})$.

Note that the image of $(\operatorname{End}_P^\beta T)_{A_0}$ under the isomorphism $\operatorname{End}_P^\beta A_0 \to H^0(Q, A_0^{(\beta)})$ of Lemma 48 is $\vartheta(H^0(Q, T^{(\beta)}))$. Hence there exists a complement E_0 of $(\operatorname{End}_P^\beta T)_{A_0}$ in $\operatorname{End}_P^\beta A_0$ which is isomorphic to $H^1(Q, T_n^{(\beta)}) \cong H^1(Q, T_n)$. Let E_β be a set of maps $T \to T$ such that $(E_\beta)_{A_0} = E_0$. Evidently, for every $k \in \mathbb{N}_0$ the group $(p^k E_\beta)_{A_k}$ is isomorphic to $H^0(Q, T_n) \cong H^0(Q, T_{n+kd})$ and intersects trivially with $(\operatorname{End}_P T)_{A_k}$. Thus $(p^k E_\beta)_{A_k}$ is a complement of $(\operatorname{End}_P^\beta T)_{A_k}$ in $\operatorname{End}_P^\beta A_k$ and the result follows. □

Recall that a and b denote $\max\{\exp H^2(P, T), \exp H^3(P, T_n)\}$ and $\exp H^1(P_{t_0}, T_n)$, respectively. By Lemma 49 there exists an additive group E of maps $T \to T$ such that $E_k := (p^k E)_{A_k}$ is a group of exponent b with $\operatorname{End}_P A_k = (\operatorname{End}_P T)_{A_k} \oplus E_k$ for every $k \geq 0$. Denote

$$\varrho_k : E_k \to \Gamma_k, \ \epsilon \mapsto (1, 1 + \epsilon).$$

Further, let ϱ and π_k be the group homomorphism defined by

$$\varrho : \begin{array}{ccc} 1 + \max\{a, b\} \cdot \operatorname{End}_P T & \to & \Gamma \\ \epsilon & \mapsto & (1, \epsilon) \end{array} \quad \text{and} \quad \pi_k : \begin{array}{ccc} \Gamma & \to & \Gamma_k \\ (\beta, \epsilon) & \mapsto & (\beta, \epsilon_{A_k}) \end{array}.$$

Lemma 50. *The map ϱ_k is a group homomorphism and $\operatorname{Im} \varrho_k$ centralizes $\operatorname{Im}(\pi_k \circ \varrho) \leq \Gamma_k$.*

Proof. Let ϵ be an element of E_k. The exponent of E_k is b and hence the image of ϵ is a subgroup of $b^{-1} T_{n+kd} / T_{n+kd}$. Let m be the floor of n/d, in particular $n - 1 \leq md \leq n$. As $\max\{a, b\} \cdot b$ is a p-power, we have $p^{n/d} \geq p^m \geq \max\{a, b\} \cdot b$ by Assumption (5.3.1). It follows by the uniserial action of P on T that $\operatorname{Im} \epsilon \leq b^{-1} T_{n+kd} / T_{n+kd} \leq b^{-1} p^m T_{kd} / T_{n+kd}$. Let ϵ' be another element of E_k. Since multiplication by p induces a P-endomorphism $A_k \to A_k$, the composition $\epsilon \circ \epsilon'$ maps from A_k to $(b^{-2} p^m T_{n+kd} + T_{n+kd}) / T_{n+kd}$. Recall that p^m is at least b^2. Hence $\epsilon \circ \epsilon'$ is trivial and $(1 + \epsilon) \circ (1 + \epsilon')$ equals $1 + \epsilon + \epsilon'$. Thus ϱ_k is a group homomorphism. It follows by similar arguments that $\operatorname{Im} \varrho_k$ acts trivially on $\operatorname{Im}(\pi_k \circ \varrho)$. □

Lemma 51. *The groups $\operatorname{Im} \varrho$ and $\operatorname{Im}(\pi_k \circ \varrho)$ are normal in Γ and Γ_k, respectively. Further, the group homomorphisms $\varrho_k : E_k \to \Gamma_k$ and $\pi_k : \Gamma \to \Gamma_k$ induce a split short exact sequence*

$$0 \to E_k \to \Gamma_k / \operatorname{Im}(\pi_k \circ \varrho) \to \Gamma / \operatorname{Im} \varrho \to 1.$$

Proof. Put $c := \max\{a, b\}$ and let σ and σ_k be the group homomorphisms $\Gamma \to \mathrm{Comp}(P, T/cT)$, $(\beta, \epsilon) \mapsto (\beta, \epsilon_{T/cT})$ and $\Gamma_k \to \mathrm{Comp}(P, T/cT)$, $(\beta, \epsilon) \mapsto (\beta, \epsilon_{T/cT})$, respectively. Evidently, the image of ϱ is the kernel of σ and hence normal in Γ. By assumption $p^{n/d}/c$ is at least b and thus Lemma 49 and Equations (5.3.5) and (5.3.6) yield $\Gamma_k = \langle \mathrm{Im}\,\pi_k, \mathrm{Im}\,\varrho_k \rangle$ and $\mathrm{Im}\,\sigma = \mathrm{Im}\,\sigma_k$. Since $\mathrm{Im}\,\varrho$ is normal in Γ and $\mathrm{Im}\,\varrho_k$ centralizes $\mathrm{Im}(\pi_k \circ \varrho)$ by Lemma 50, the image $\mathrm{Im}(\pi_k \circ \varrho)$ is normalized by $\mathrm{Im}\,\pi_k$ and hence normal in $\Gamma_k = \langle \mathrm{Im}\,\pi_k, \mathrm{Im}\,\varrho_k \rangle$. Now it is straightforward to deduce from $\mathrm{Im}\,\sigma = \mathrm{Im}\,\sigma_k$ that ϱ_k and π_k induce a split short exact sequence. $\square$

Lemma 51 enables us to prove Theorem 45. First, note that for every $k \geq 0$ the p-power $a = \max\{\exp H^2(P, T), \exp H^3(P, T_n)\}$ is the exponent of $H^2(P, A_k) \cong H^2(P, T) \oplus H^3(P, T_{n+kd})$ by Short Exact Sequence (5.3.3). It follows that the action of $\mathrm{Im}(\pi_k \circ \varrho) \leq \{(1, 1 + a \cdot \varphi_{A_k}) \mid \varphi \in \mathrm{End}_P\, T\}$ on $H^2(P, A_k)$ is trivial. Thus, in order to prove Theorem 45 it suffices to consider the $\Gamma_k/\mathrm{Im}(\pi_k \circ \varrho)$-orbits of $H^2(P, A_k)$. By Lemma 51 the group $\Gamma_k/\mathrm{Im}(\pi_k \circ \varrho)$ is naturally isomorphic to $\Gamma/\mathrm{Im}\,\varrho \ltimes E_k$. Hence there exists a unique group isomorphism λ_k such that the following diagram commutes,

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d] & \Gamma_0/\mathrm{Im}(\pi_0 \circ \varrho) \arrow[d, "\lambda_k"'] & \Gamma/\mathrm{Im}\,\varrho \arrow[l] \arrow[d, equal, "\mathrm{id}"'] \\
E_k \arrow[r] & \Gamma_k/\mathrm{Im}(\pi_k \circ \varrho) & \Gamma/\mathrm{Im}\,\varrho \arrow[l]
\end{tikzcd}$$

where the horizontal arrows are induced by ϱ_0, ϱ_k, π_0 and π_k, and $E_0 \to E_k$ maps ϵ_{A_0} to $p^k \epsilon_{A_k}$. Further, let $\mu_k : H^2(P, A_0) \to H^2(P, A_k)$ be as in Section 5.3.

Lemma 52. *For $k \geq 0$, $\tau \in H^2(P, A_0)$ and $g \in \Gamma_0/\mathrm{Im}(\pi_0 \circ \varrho)$ we have $\mu_k(\tau.g) = \mu_k(\tau).\lambda_k(g)$. In particular μ_k induces a bijection from Γ_0-orbits to Γ_k-orbits.*

Proof. Recall that E is a set of maps $T \to T$ with $E_0 = E_{A_0}$ and $E_k = (p^k E)_{A_k}$. It suffices to show that for $\epsilon \in E$ the image of $\tau.(1 + \epsilon_{A_0})$ under μ_k equals $\mu_k(\tau).(1 + p^k \epsilon_{A_k})$. Let $\delta'_{2,0} : H^3(P, T_n) \to Z^2(P, A_0)$ be the group monomorphism introduced in Section 5.3 and recall that $\delta'_{2,0}$ makes the Short Exact Sequence (5.3.2) split. Then there exist $\gamma \in Z^2(P, T)$ and $\delta \in \mathrm{Im}\,\delta'_{2,0}$ such that

$$\tau = \gamma_{A_0} + \delta_{A_0} + B^2(P, A_0) \text{ and } \mu_k(\tau) = \gamma_{A_k} + p^k \delta_{A_k} + B^2(P, A_k),$$

see Equation (5.3.4). Further, recall that $a \geq \exp \mathrm{Im}\,\delta'_{2,0}$ and $b \geq \exp E_0$. This yields $\mathrm{Im}\,\delta_{A_0} \leq a^{-1} T_n/T_n$ and $\mathrm{Im}\,\epsilon_{A_0} \leq b^{-1} T_n/T_n$. Let m be the floor of n/d, in particular $n - 1 \leq md \leq n$. As $\max\{a, b\} \cdot b$ is a p-power, the inequality $p^{n/d} \geq p^m \geq \max\{a, b\} \cdot b$ holds. Since T_n is a subgroup of $T_{md} = p^m T$, we have $\mathrm{Im}\,\epsilon_{A_0} \leq p^m b^{-1} T/T_n \leq aT/T_n$. It follows that $\mathrm{Im}(\delta_{A_0}.\epsilon_{A_0})$ is a subgroup of $(p^m a^{-1} b^{-1} T_n + T_n)/T_n$ and hence $\delta_{A_0}.\epsilon_{A_0}$ and $\delta_{A_k}.(p^k \epsilon_{A_k})$ are trivial. This yields

$$\tau.(1 + \epsilon_{A_0}) = \tau + \gamma_{A_0}.\epsilon_{A_0} + B^2(P, A_0) \text{ and } \mu_k(\tau).(1 + p^k \epsilon_{A_k}) = \mu_k(\tau) + \gamma_{A_k}.(p^k \epsilon_{A_k}) + B^2(P, A_k).$$

Since $\mathrm{Im}\,\epsilon_{A_0} \leq aT/T_n$ and $a \geq \exp H^2(P, T)$, we have further that $\gamma_{A_0}.\epsilon_{A_0}$ lies in $\mathrm{Im}\,\delta'_{2,0} + B^2(P, A_0)$. By Equation (5.3.4) the image $\mu_k(\gamma_{A_0}.\epsilon_{A_0} + B^2(P, A_0))$ equals $\gamma_{A_k}.(p^k \epsilon_{A_k}) + B^2(P, A_k)$. The result follows. $\square$

Proof of Theorems 45 and 43. Theorem 45 is a direct consequence of Lemma 52. Recall that Theorem 43 is equivalent to Theorem 45 by Theorem 44, Lemma 39 and Equation (5.3.4). $\square$

Lemma 52 yields

Corollary 53. *For $k \geq 0$ and $\tau \in H^2(P, A_0)$ the image of the stabilizer* $\mathrm{Stab}_{\Gamma_0}(\tau)/\mathrm{Im}(\pi_0 \circ \varrho)$ *under λ_k equals* $\mathrm{Stab}_{\Gamma_k}(\mu_k(\tau))/\mathrm{Im}(\pi_k \circ \varrho)$.

The last Corollary 53 will be applied in order to determine the automorphism groups of coclass families.

5.4 Shaved coclass graphs and coclass trees

In this section we consider the structure of the coclass graph $\mathcal{G}(p, r)$, where p and r denote a prime and a natural number, respectively. For this purpose, we recall some notation for the graph $\mathcal{G}(p, r)$ given in [9]. Let G and H be two p-groups of coclass r. Then G is called a *descendant* of H, if they represent the same vertex or if there is a path from H to G. The *descendant tree* of H is defined as the induced subgraph of $\mathcal{G}(p, r)$ whose vertex set consists of the descendants of H. Further, we say that a group G has *distance* k from a group H if G is a descendant of H and the length of the path from H to G is k. This implies that $|G| = |H| \cdot p^k$. The groups of distance 1 from H are called *immediate descendants* of H.

Definition 54. For a natural number m the *shaved coclass graph* $\mathcal{G}_m(p, r)$ is the subgraph of $\mathcal{G}(p, r)$ induced by the groups having distance at most m from a mainline group of $\mathcal{G}(p, r)$. The groups in $\mathcal{G}_m(p, r)$ which are not mainline groups are called *off-mainline groups*.

The coclass graph $\mathcal{G}(2, 2)$ and the shaved coclass graph $\mathcal{G}_1(2, 2)$ are displayed in Figures 5.2 and 5.3, respectively. The mainlines of $\mathcal{G}(2, 2)$ are colored green in Figure 5.2.
In what follows, let S be an infinite pro-p-group of coclass r and let $S/\gamma_e(S)$ be the root of the corresponding mainline.

Definition 55. The *coclass tree* $\mathcal{T}(S)$ is the descendant tree of $S/\gamma_e(S)$ in the coclass graph $\mathcal{G}(p, r)$. For $i \geq e$ the i-th *branch* of the coclass tree $\mathcal{T}(S)$, denoted by $\mathcal{B}_i(S)$, is defined as the subgraph of $\mathcal{G}(p, r)$ induced by the descendants of $S/\gamma_i(S)$ which are not descendants of $S/\gamma_{i+1}(S)$.

For $m \in \mathbb{N}$ the *shaved coclass tree* $\mathcal{T}_m(S)$ is defined similar to the coclass tree $\mathcal{T}(S)$, that is, $\mathcal{T}_m(S) := \mathcal{T}(S) \cap \mathcal{G}_m(p, r)$. Further, for $i \geq e$ we write $\mathcal{B}_{m,i}(S) := \mathcal{B}_i(S) \cap \mathcal{T}_m(S)$ to denote the i-th branch of $\mathcal{T}_m(S)$.
Du Sautoy [5] has shown that for fixed m the shaved coclass tree $\mathcal{T}_m(S)$ is ultimately periodic; that is, apart from a finite piece at the top the shaved coclass tree is periodic. In the remainder of this section we outline a constructive proof for the ultimate periodicity given by Eick & Leedham-Green [9]. We shall see in Theorem 59 that all but finitely many groups in $\mathcal{T}_m(S)$ are elements of coclass families and that coclass families are closely related to the structure of shaved coclass trees.

Theorem 56 ([9, Theorem 7])**.** *Let m be a natural number. Then there exists a natural number $f = f(m, S) \geq e$ such that*

(i) *the group $T := \gamma_f(S)$ is a translation subgroup of S and $P := S/\gamma_f(S)$ is a mainline group acting uniserially on T with lower P-central series $T = T_0 \leq T_1 \leq T_2 \leq \cdots$, say, and*

(ii) *every descendant G of P in $\mathcal{G}_m(p, r)$ is a group extension of T/T_i by P, where i is the p-logarithm of $|G|/|P|$.*

For the remainder of this section, we fix $m \in \mathbb{N}$. Further, let P, T and f be as in Theorem 56 and let d be the finite rank of the free $\mathbb{Z}_p$-module T. Recall that the uniserial action of P on T

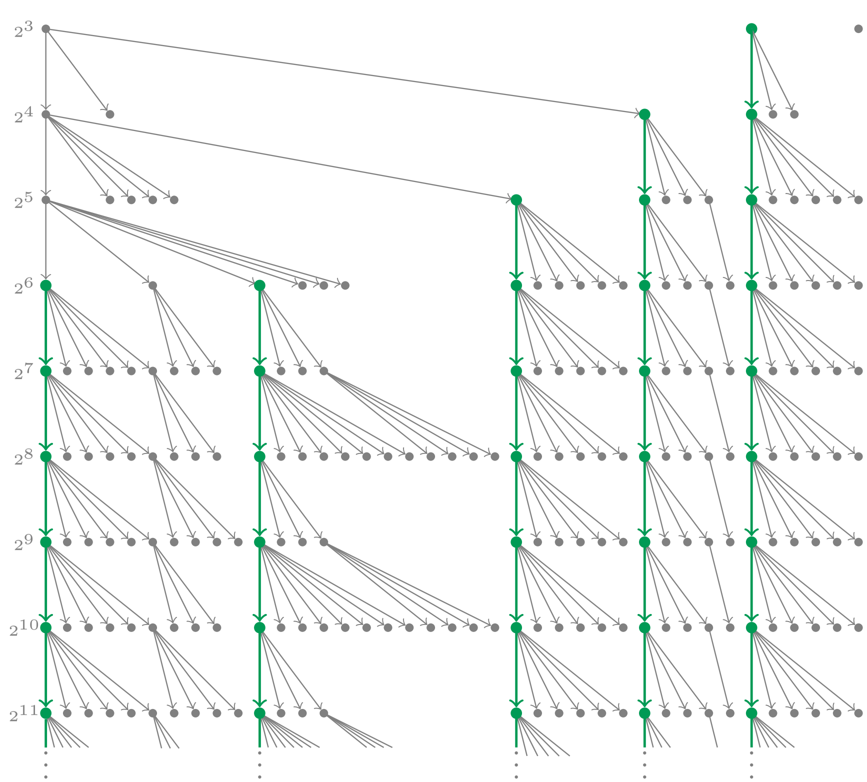


Figure 5.2: Coclass graph $\mathcal{G}(2,2)$

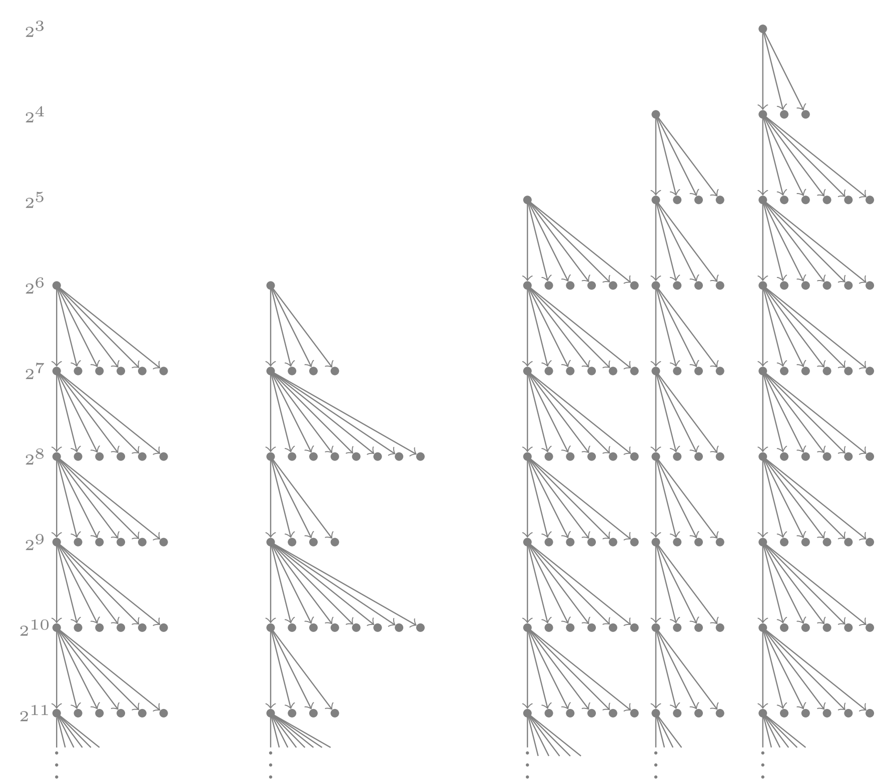


Figure 5.3: Shaved coclass graph $\mathcal{G}_1(2,2)$

yields $T_{i+d} = p \cdot T_i$. By Theorem 56 every descendant G of P can be represented by an element of $H^2(P, T/T_n)$ for suitable n. Further, recall that for a 2-cochain $\tau \in C^2(P,T)$ and $n \in \mathbb{N}$ we write τ_{T/T_n} to denote the composition of τ and the natural homomorphism $T \twoheadrightarrow T/T_n$. As in Section 5.3 let $t_0 \in T$ be an element with $\langle t_0, T_1 \rangle = T$, and for a natural number n let $a(n)$ and $b(n)$ denote $\max\{\exp H^2(P,T), \exp H^3(P,T_n)\}$ and $\exp H^1(P_{t_0}, T_n)$, respectively. Note that $a(n+d) = a(n)$ and $b(n+d) = b(n)$ by the uniserial action of P on T. Let v be the natural number with

$$p^v = \max\{b(n) \cdot \max\{a(n), b(n)\} \mid 0 \leq n < d\}.$$

For a natural number $n \geq vd$ and for every nonnegative integer k let the group isomorphisms $\mu(n)_k : H^2(P, T/T_n) \to H^2(P, T/T_{n+kd})$ be constructed as in Section 5.3. Recall there exists a subgroup $A(n) \leq C^2(P, T_{n-\log_p(a(n))d})$ such that $Z^2(P,T/T_n) = Z^2(P,T)_{T/T_n} \oplus A(n)_{T/T_n}$, and for $\tau = \delta_{T/T_n} + \epsilon_{T/T_n} \in Z^2(P, T/T_n)$ with $\delta \in Z^2(P,T)$ and $\epsilon \in A(n)$ we have

$$\mu(n)_k(\tau + B^2(P,T/T_n)) = \delta_{T/T_{n+kd}} + p^k \epsilon_{T/T_{n+kd}} + B^2(P, T/T_{n+kd}) \tag{5.4.7}$$

for every $k \in \mathbb{N}_0$, see Equation (5.3.4). We write μ_* to denote the map defined by setting $\mu_*(\tau) = \mu(n)_1(\tau)$ for $\tau \in H^2(P, T/T_n)$ with $n \geq vd$. We may assume that $A(o) \leq A(n)$ for $o \geq n \geq vd$, in particular for $\tau \in H^2(P,T/T_n)$ and $k \in \mathbb{N}_0$ we have

$$\mu_*(\tau_{T/T_o}) = \mu_*(\tau)_{T/T_{o+d}} \text{ and } \mu_*^k(\tau) = \mu(n)_k(\tau),$$

where μ_*^k denotes the k-th iterate of μ_*.
In what follows, we construct a graph isomorphism from the branch $\mathcal{B}_{m,i}$ to $\mathcal{B}_{m,i+d}$ for suitable i. This construction goes back to Eick & Leedham-Green [9]. The proof in [9] showing that the graph isomorphism is well-defined is incorrect as a consequence of the above mentioned mistake in [9, Theorem 23] and has to be corrected. For this purpose we need the following lemmata.

Lemma 57. *Let n be a natural number with $n \geq vd$. Further, let $\tau \in Z^2(P,T/T_n)$ and $\tau_* \in Z^2(P, T/T_{n+d})$ be cocycles with $\mu_*(\tau + B^2(P,T/T_n)) = \tau_* + B^2(P, T/T_{n+d})$. Then $E(\tau)$ has coclass r if and only if this is the case for $E(\tau_*)$.*

Proof. Let l be the nilpotency class of P. Note that a group extension $E(\lambda)$ defined by a cocycle $\lambda \in Z^2(P, T/T_m)$ with $m \in \mathbb{N}_0$ has coclass $r = \operatorname{cc} P$ if and only if $\gamma_l(E(\lambda)) = T/T_m$, where $\gamma_l(E(\lambda))$ denotes the l-th term of the lower central series of $E(\lambda)$.
Recall that $a(n) \geq \exp H^3(P,T_n)$. Hence by construction of μ_* there exists $\gamma \in Z^2(P,T)$ and $\epsilon \in C^2(P, T_{n-\log_p(a(n))d})$ with $\gamma_{T/T_n} + \epsilon_{T/T_n} = \tau$ and $\gamma_{T/T_{n+d}} + p\epsilon_{T/T_{n+d}}$ equals τ_* modulo $B^2(P, T/T_{n+d})$. Since $n \geq vd > \log_p(a(n))d$, we have $\epsilon_{T/T_d} = 0$. It follows that $\gamma_{T/T_d} = \tau_{T/T_d} = (\tau_*)_{T/T_d}$. Then for $\lambda \in \{\tau, \tau_*\}$ the group extension $E(\lambda)$ has coclass r if and only if $\gamma_l(E(\gamma_{T/T_d})) = T/T_d$ holds. The result follows. □

Lemma 58. *Let n be a natural number with $n \geq vd$. Then μ_* induces a bijection between the isomorphism classes of group extensions of T/T_n by P of coclass r and the isomorphism classes of group extensions of T/T_{n+d} by P of coclass r.*

Proof. Let $\gamma_1, \gamma_2 \in Z^2(P, T/T_n)$ be such that $E(\gamma_1)$ and $E(\gamma_2)$ have coclass r and $E(\gamma_1)$ is isomorphic to $E(\gamma_2)$. Then by Theorem 44 the elements $\gamma_1 + B^2(P,T/T_n)$ and $\gamma_2 + B^2(P,T/T_n)$ lie in the same orbit under the action of $\operatorname{Comp}(P,T/T_n)$. By Theorem 45 the map μ_* induces a bijection between the orbits of $H^2(P,T/T_n)$ under the action of $\operatorname{Comp}(P,T/T_n)$ and the orbits of $H^2(P, T/T_{n+d})$ under the action of $\operatorname{Comp}(P, T/T_{n+d})$. The result follows by Lemma 57. □

Lemma 58 enables us to construct graph isomorphisms between the branches of the shaved coclass tree $\mathcal{T}_m(S)$. For a p-group G of coclass r let $\mathrm{V}(G)$ denote the vertex in $\mathcal{G}(p,r)$ corresponding to the isomorphism class of G. Further, for a cocycle $\gamma \in Z^2(P, T/T_l)$ with $l \in \mathbb{N}$ and $\operatorname{cc} E(\gamma) = r$ let $\mathrm{V}(\gamma)$ denote $\mathrm{V}(E(\gamma))$.
Let f be as in Theorem 56. In particular T equals $\gamma_f(S)$. For $i \geq vd + f$ we write ν_i to denote the map $\mathcal{B}_{m,i} \to \mathcal{B}_{m,i+d}$ induced by μ_*, that is, for a cocycle $\gamma \in Z^2(P, T_k)$ with $k \in \mathbb{N}$ and $\mathrm{V}(\gamma) \in \mathcal{B}_{m,i}$ the vertex $\mathrm{V}(\gamma)$ is mapped to $\mathrm{V}(\gamma_*)$, where γ_* is a representative of $\mu_*(\gamma + B^2(P, T/T_k))$.

Theorem 59. *Let $i \in \mathbb{N}$ be such that $i - f \geq vd$. Then ν_i is a well-defined graph isomorphism.*

Proof. Denote $\nu := \nu_i$ and note that by Lemma 58 the image of a vertex under ν does not depend on the chosen cocycle.
Let G and H be groups in $\mathcal{B}_{m,i}$ such that H is an immediate descendant of G. Note that G and H are descendants of $S/\gamma_i(S)$. Then there are $k \in \mathbb{N}$ and $\tau \in Z^2(P, T/T_k)$ such that $\mathrm{V}(G) = \mathrm{V}(\tau)$, $\mathrm{V}(H) = \mathrm{V}(\tau_{T/T_{k-1}})$ and $\mathrm{V}(\tau_{T/T_{i-f}}) = \mathrm{V}(S/\gamma_i(S))$.
Recall that by Equation (5.4.7) there exist $\delta \in Z^2(P,T)$ and $\epsilon \in C^2(P, T_{k-\log_p(a(k))d})$ such that $\tau = \delta_{T/T_k} + \epsilon_{T/T_k}$ and

$$\mu_*(\tau_{T/T_l} + B^2(P, T/T_l)) = \delta_{T/T_{l+d}} + p\epsilon_{T/T_{l+d}} + B^2(P, T/T_{l+d})$$

for $k \geq l \geq i - f$. If follows that $\nu(V(G))$ is an immediate descendant of $\nu(V(H))$. Since $k \geq vd > \log_p(a(k))d$, we have $\tau_{T/T_d} = \delta_{T/T_d}$ and hence G is a descendant of $E(\delta_{T/T_d})$ and $E(\delta_{T/T_d})$ lies in the coclass tree $\mathcal{T}(S)$. Further, $E(\delta)$ is a pro-p-group of coclass r. As every coclass tree corresponds to exactly one isomorphism class of an infinite pro-p-group of coclass r by definition, the group $E(\delta)$ is isomorphic to S. Thus $E(\delta_{T/T_{i-f}}) \cong S/\gamma_i(S)$ and $E(\delta_{T/T_{i-f+d}}) \cong S/\gamma_{i+d}(S)$. This yields that $\nu(V(G))$ is a descendant of $\nu(V(S/\gamma_i(S))) = V(S/\gamma_{i+d}(S))$ and hence $\operatorname{Im}\nu \subseteq \mathcal{B}_{m,i+d}$. It follows that ν is a graph isomorhism.
Now, we show that ν is a bijection. For this purpose let $\nu^{-1} : \mathcal{B}_{i+d} \to \mathcal{B}_i$ be the graph homomorphism induced by $(\mu_*)^{-1}$. Similarly to above, we can show that ν^{-1} is a well-defined graph homomorphism. Evidently ν^{-1} is an inverse of ν and thus ν is a bijection. $\square$

We write ν_* to denote the map defined by setting $\nu_*(V(G)) := \nu_i(V(G))$ for $V(G) \in \mathcal{B}_{m,i}$ with $i \geq vd + f$.
A coclass family $\mathcal{F}(\gamma, \delta, n) = (G_k \mid k \geq 0)$ with associated point group P, translation subgroup T and pro-p-group $E(\gamma) \cong S$ is called a *strict coclass family* if $V(G_0)$ is a vertex in $\mathcal{B}_{m,i}$ for some $i \geq vd + f$ and $\nu_*(V(G_k)) = V(G_{k+1})$ for every $k \geq 0$. Evidently, every strict coclass family is a coclass family. It is straightforward to show that every coclass family $\mathcal{F}(\gamma, \delta, n)$ with $\gamma \in Z^2(P,T)$ and $E(\gamma) \cong S$ is isomorphic to some strict coclass family. The definition of strict coclass families is equivalent to the one of coclass sequences in [10].
We illustrate the results of this section by considering a shaved coclass tree in $\mathcal{G}_2(2,2)$ as an example. Let P be the group given by the pc-presentation $\langle g_1, \ldots, g_5 \mid \Omega \rangle$, where Ω consists of

$$\begin{array}{lllll}
\omega_{1,1} := g_1^2 g_4, & \omega_{2,2} := g_2^2, & \omega_{3,3} := g_3^2, & \omega_{4,4} := g_4^2, & \omega_{5,5} := g_5^2 \\
\omega_{1,2} := g_2^{g_1} g_2 g_3, & \omega_{1,3} := g_3^{g_1} g_3 g_5, & \omega_{1,4} := g_4^{g_1} g_4, & \omega_{1,5} := g_5^{g_1} g_5, & \omega_{2,3} := g_3^{g_2} g_3, \\
\omega_{2,4} := g_4^{g_2} g_4 g_5, & \omega_{2,5} := g_5^{g_2} g_5, & \omega_{3,4} := g_4^{g_3} g_4, & \omega_{3,5} := g_5^{g_3} g_5, & \omega_{4,5} := g_5^{g_4} g_5,
\end{array}$$

and let P act uniserially on a free $\mathbb{Z}_p$-module $T = \langle t_1, t_2 \rangle$ of rank 2 via the group homomorphism $\mathfrak{X} : P \to \mathrm{GL}_2(\mathbb{Z}_p)$ with values

$$\mathfrak{X}(g_1) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \mathfrak{X}(g_2) = \mathfrak{X}(g_4) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathfrak{X}(g_3) = \mathfrak{X}(g_5) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a 2-cocycle γ let δ_γ be defined as in Section 4.2. By [10, Section B.1] there exists a cocycle $\gamma \in Z^2(P,T)$ such that δ_γ takes the values in Table 5.1, the group extension $S := E(\gamma)$ is a pro-2-group of coclass 2 and $S/\gamma_3(S) \cong P$ is a mainline group in $\mathcal{G}_2(2,2)$. In our definition of S we substituted t_1 and t_2 in the group presentation given in [10, Section B.1] by t_2 and t_1, respectively. Let

$$T = T_0 > T_1 > T_2 > \cdots$$

be the lower P-central series of T and note that $T_1 = \langle 2t_1, t_2\rangle$ and T_{n+2} is $2 \cdot T_n$ for $n \in \mathbb{N}_0$. Then by [10, Section B.1, coclass sequences $\mathcal{K}^6$ and $\mathcal{K}^{16}$] there exist 2-cochain maps $\alpha, \beta \in C^2(P,T)$ such that $\alpha_{T/T_2} \in Z^2(P,T/T_2)$ and $\beta_{T/T_3} \in Z^2(P,T/T_3)$ are cocycles defining coclass families $(G_k \mid k \geq 0) := \mathcal{F}(\gamma,\alpha,2) = (E(\gamma_{T/T_{2+2\cdot k}} + 2^k \cdot \alpha_{T/T_{2+2\cdot k}}) \mid k \geq 0)$ and $(H_k \mid k \geq 0) := \mathcal{F}(\gamma,\beta,3) = (E(\gamma_{T/T_{3+2\cdot k}} + 2^k \beta_{T/T_{3+2\cdot k}}) \mid k \geq 0)$ in $\mathcal{G}_2(2,2)$, and $\delta_{\alpha_{T/T_2}}$ and $\delta_{\beta_{T/T_3}}$ take the values given in Table 5.1. The shaved coclass tree induced by the pro-p-group S in $\mathcal{G}_2(2,2)$ and the coclass families

ϵ	$\delta_\epsilon(\omega_{1,1})$	$\delta_\epsilon(\omega_{2,2})$	$\delta_\epsilon(\omega_{3,3})$	$\delta_\epsilon(\omega_{4,4})$	$\delta_\epsilon(\omega_{5,5})$	$\delta_\epsilon(\omega_{1,2})$	$\delta_\epsilon(\omega_{1,3})$	$\delta_\epsilon(\omega_{1,4})$
γ	0	0	t_1+t_2	0	$-t_2$	0	t_1	0
α_{T/T_2}	0	$-t_2$	0	0	0	$-t_2$	0	0
β_{T/T_3}	$2t_1$	$-t_2$	$2t_1$	$2t_1$	0	$2t_1-t_2$	$2t_1$	$2t_1$
ϵ	$\delta_\epsilon(\omega_{1,5})$	$\delta_\epsilon(\omega_{2,3})$	$\delta_\epsilon(\omega_{2,4})$	$\delta_\epsilon(\omega_{2,5})$	$\delta_\epsilon(\omega_{3,4})$	$\delta_\epsilon(\omega_{3,5})$	$\delta_\epsilon(\omega_{4,5})$	
γ	$-t_1-t_2$	0	t_1	0	$-t_1-t_2$	$-t_2$	0	
α_{T/T_2}	0	0	0	0	0	0	0	
β_{T/T_3}	0	$2t_1$	0	0	$2t_1$	0	0	

Table 5.1

$(G_k \mid k \geq 0)$ and $(H_k \mid k \geq 0)$ are pictured in Figure 5.4 on page 42.

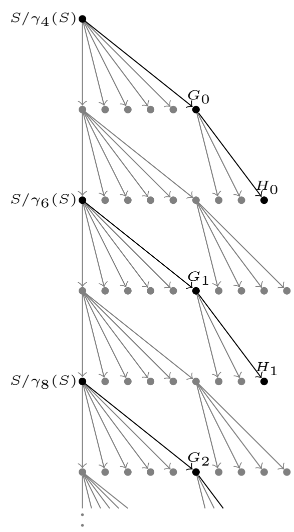


Figure 5.4: Some coclass families in $\mathcal{G}_2(2,2)$

Chapter 6

Parametrized families

In Part II of this thesis we consider irreducible characters of groups in a coclass family $\mathcal{F} = \mathcal{F}(\gamma, \delta, n) = (G_k \mid k \geq 0)$, say. This requires a detailed analysis of the groups G_k and their subgroups. For this purpose, we use ideas from the theory of polycyclic groups.

Let P, T and S be the associated point group, translation subgroup and pro-p-group of the coclass family $\mathcal{F}(\gamma, \delta, n)$. In particular γ lies in $Z^2(P, T)$, the group P is a finite p-group acting uniserially on the free $\mathbb{Z}_p$-module T of finite rank $d \in \mathbb{N}$ and S is the group extension $E(\gamma)$. Further, let

$$T = T_0 \geq T_1 \geq T_2 \geq \cdots$$

be the lower P-central series of T and denote $A_k := T/T_{n+kd}$. By definition G_k is an extension of $A_k = T/T_{n+kd}$ by P defined by the cocycle $\gamma_{A_k} + p^k \delta_{A_k}$. Recall that we write the elements of G_k in the form $(g, t + T_{n+kd})$ with $g \in P$ and $t \in T$.

By abuse of notation, let T and A_k also denote the images of the natural embeddings $T \hookrightarrow S$, $t \mapsto (1, t)$ and $A_k \hookrightarrow G_k = E(\gamma_{A_k} + p^k \delta_{A_k})$, $t + T_{n+kd} \mapsto (1, t + T_{n+kd})$. To avoid any confusion we write T_+ instead of T, if we consider T as $\mathbb{Z}_pP$-module. Note that the $\mathbb{Z}_pP$-module T_+ and the subgroup $T \leq S$ are written additively and multiplicatively, respectively.

By Lemma 40 we may assume that $n \in d \cdot \mathbb{N}$. Denote $n' := n/d$ and note that $T_{n+kd} = p^{n'+d} \cdot T_+$ by the uniserial action of P on T_+.

6.1 Generic polycyclic sequences

In this section we describe polycyclic sequences for the groups $G_k = E(\gamma_{A_k} + p^k \delta_{A_k})$. Let $\{t_1, \ldots, t_d\}$ be a generating set of T_+ and let $(g_1, \ldots, g_q)$ be a polycyclic sequence of P. Recall that $S = E(\gamma)$ and for $1 \leq i \leq q$, $1 \leq j \leq d$ and $k \in \mathbb{N}_0$ define

$$\begin{array}{ll} x_i & := (g_i, 0) \in S, \\ x_{q+j} & := (1, t_j) \in S \end{array} \quad \text{and} \quad \begin{array}{ll} x_{i,k} & := (g_i, 0 + T_{n+kd}) \in G_k, \\ x_{q+j,k} & := (1, t_j + T_{n+kd}) \in G_k. \end{array}$$

Denote $G_k(j) := \langle x_{j,k}, \ldots, x_{q+d,k} \rangle$ for $1 \leq j \leq q + d$ and $G_k(q + d + 1) := 1$. Then the series $(G_k(j) \mid 1 \leq j \leq q + d + 1)$ is a polycyclic series of G_k with corresponding polycyclic sequence $X_k := (x_{1,k}, \ldots, x_{q+d,k})$. Denote $X_{A_k} := (x_{q+1,k}, \ldots, x_{q+d,k})$ and note that X_{A_k} is an induced polycyclic sequence of $A_k \leq G_k$.

Further let $X_T := (x_{q+1}, \dots, x_{q+d})$. We define an exponent function and a leading exponent function for X_T similar to Section 4.1 allowing images in the p-adic numbers $\mathbb{Z}_p$ instead of $\mathbb{Z}$:

$$\begin{aligned} &\exp_{X_T} : T \to \mathbb{Z}_p^d, && \prod_{i=1}^d x_{q+i}^{a_i} \mapsto (a_1, \dots, a_d),\\ &\mathrm{lead}_{X_T} : T \setminus 1 \to \mathbb{Z}_p, && t \mapsto a_t, \end{aligned}$$

where a_t is the first nontrivial entry of $\exp_{X_T}(t)$. As T and A_k are abelian, the exponent functions $\exp_{X_T} : T \to \mathbb{Z}_p^d$ and $\exp_{X_{A_k}} : A_k \to (\mathbb{Z}/p^{n'+k}\mathbb{Z})^d$ are group isomorphisms.

6.2 Parametrized families of subgroups

We now introduce certain families of subgroups $(U_k \mid k \geq 0)$ with $U_k \leq G_k$ for each $k \geq 0$. These subgroup families are defined in a way that allows to work with all subgroups in the family simultaneously.
As a first step, we define generic generating sets for these subgroups.

Definition 60. Let η_k and κ_k denote the surjection $S \to G_k$, $(g,t) \mapsto (g, t + T_{n+kd})$ and the natural homomorphism $G_k \twoheadrightarrow P \cong G_k/A_k$, $(g, t + T_{n+kd}) \mapsto g$, respectively.

(i) Let $a \in \mathbb{N}$, $v = (v_1, \dots, v_a) \in S^a$ and $w = (w_1, \dots, w_a) \in T^a$. Then for $c \in \mathbb{N}_0$ define a sequence $v_k = (v_{1,k}, \dots, v_{a,k}) \in G_k^a$ for $k \in \mathbb{N}_0$ via

$$v_{i,k} = \begin{cases} \eta_k(v_i), & \text{if } k < c,\\ \eta_k(v_i) \cdot \eta_k(w_i)^{p^{k-c}}, & \text{if } k \geq c. \end{cases}$$

Let $\mathcal{E}_c(v,w) = (v_{1,k}, \dots, v_{a,k} \mid k \geq 0)$ be the *strict generators sequence* defined by the nonnegative integer c and the sequences v and w.

(ii) Let $a \in \mathbb{N}$ and $\mathcal{E} = (v_{1,k}, \dots, v_{a,k} \mid k \geq 0)$ be such that $v_{i,k} \in G_k$ and $\kappa_k(v_{i,k}) = \kappa_0(v_{i,0})$. Then $\mathcal{E}$ is called *generators sequence* if $\mathcal{E}$ differs from a strict generators sequence only in finitely many elements.

(iii) A family $(V_k \mid k \geq 0)$ of subgroups $V_k \leq G_k$ is called *parametrized*, if there exists a generators sequence $(v_{1,k}, \dots, v_{a,k} \mid k \geq 0)$ with $V_k = \langle v_{1,k}, \dots, v_{a,k} \rangle$.

Note that $\eta_k : S \to G_k$ is in general not a group homomorphism since S is a group extension defined by $\gamma \in Z^2(P,T)$ whereas G_k is defined by $\gamma_{A_k} + p^k \delta_{A_k} \in Z^2(P, A_k)$.
In Subsections 6.3 and 6.4 we collect some features of parametrized families of subgroups. Here, we summarize the main results of these subsections. For the proofs of the statements we refer to the corresponding statements, which are stated in brackets.

Lemma 61 (Lemma 63)**.** *Let $(v_{1,k}, \dots, v_{a,k} \mid k \geq 0) = \mathcal{E}_c(v,w)$ be a strict generators sequence and let ω be a word in the free group of rank a. Then there exist some $v' \in S$ and $w' \in T$ such that $\mathcal{E}_c(v', w') = (\omega(v_{1,k}, \dots, v_{a,k}) \mid k \geq 0)$. In particular $(\omega(v_{1,k}, \dots, v_{a,k}) \mid k \geq 0)$ is a strict generators sequence.*

Theorem 62 (Proposition 72, Theorem 78 & Lemma 71)**.** *Let $(U_k \mid k \geq 0)$ and $(V_k \mid k \geq 0)$ be parametrized. Then*

(i) *the families $(U_k' \mid k \geq 0)$ and $(U_k \cap V_k \mid k \geq 0)$ are parametrized,*

(ii) *there exist $j \in \{0, \dots, d\}$ and $\alpha \in \mathbb{Q}$ such that $[G_k : U_k] = \alpha p^{jk}$ for all but finitely many k.*

6.3 Characterization of parametrized families

In this section and in Section 6.4 we prove Lemma 61 and Theorem 62. For this purpose, products and inverses of elements in $G_k = E(\gamma_{A_k} + p^k\delta_{A_k})$ are considered for large k. For elements $(g, t + T_{n+kd})$ and $(h, s + T_{n+kd})$ in G_k we have

$$(g, t + T_{n+kd})(h, s + T_{n+kd}) = (gh, \gamma(g, h) + p^k\delta(g, h) + t.h + s + T_{n+kd}), \qquad (6.3.1)$$
$$(g, t + T_{n+kd})^{-1} = (g^{-1}, -\gamma(g, g^{-1}) - p^k\delta(g, g^{-1}) - t.g^{-1} + T_{n+kd}) \qquad (6.3.2)$$

by [26, p. 316f.].

Lemma 63. *Let $(v_{1,k}, \dots, v_{a,k} \mid k \geq 0) = \mathcal{E}_c(v, w)$ be a strict generators sequence and let ω be a word in the free group of rank a. Then there exist some $v' \in S$ and $w' \in T$ such that $\mathcal{E}_c(v', w') = (\omega(v_{1,k}, \dots, v_{a,k}) \mid k \geq 0)$. In particular $(\omega(v_{1,k}, \dots, v_{a,k}) \mid k \geq 0)$ is a strict generators sequence.*

Proof. It suffices to show the claim for $(v_{1,k}^{-1} \mid k \geq 0)$ and $(v_{1,k} \cdot v_{2,k} \mid k \geq 0)$. Let $v_1, v_2 \in P$ and $\tilde{v}_1, \tilde{v}_2, w_1, w_2 \in T_+$ be such that $v_{i,k} = (v_i, \tilde{v}_i + p^{k-c}w_i + T_{n+kd})$ for $i \in \{1, 2\}$ and $k \geq c$. Equations (6.3.1) and (6.3.2) yield

$$v_{1,k}^{-1} = (v_1^{-1}, -\gamma(v_1, v_1^{-1}) - \tilde{v}_1.v_1^{-1} + p^{k-c}(-p^c \cdot \delta(v_1, v_1^{-1}) - w_1.v_1^{-1}) + T_{n+kd}) \text{ and}$$

$$v_{1,k} \cdot v_{2,k} = (v_1 \cdot v_2, \gamma(v_1, v_2) + \tilde{v}_1.v_2 + \tilde{v}_2 + p^{k-c}(w_1.v_2 + p^c \cdot \delta(v_1, v_2) + w_2) + T_{n+kd}).$$

Put $v' := (v_1^{-1}, -\gamma(v_1, v_1^{-1}) - \tilde{v}_1.v_1^{-1}) \in S$ and $w' := (0, -p^c \cdot \delta(v_1, v_1^{-1}) - w_1.v_1^{-1})$. Then we have $\mathcal{E}_c(v', w') = (v_{1,k}^{-1} \mid k \geq 0)$. Similarly, we find $v'' \in S$ and $w'' \in T$ satisfying $\mathcal{E}_c(v'', w'') = (v_{1,k} \cdot v_{2,k} \mid k \geq 0)$. The result follows. ☐

We show in several steps that a family $\mathcal{U}$ is parametrized if and only if there exists a generators sequence consisting of induced polycyclic sequences generating $\mathcal{U}$. Our efforts will result in Theorem 70.

Lemma 64. *Let $(U_k \mid k \geq 0)$ be parametrized. Then there exist generators sequences $(Z_k \mid k \geq 0)$ and $(Y_k \mid k \geq 0)$ such that Z_k and Y_k have entries in U_k, the image of Z_k under the natural homomorphism $G_k \twoheadrightarrow G_k/A_k$ is an induced polycyclic sequence of U_kA_k/A_k w.r.t. $(x_{1,k}A_k, \dots, x_{q,k}A_k)$ and the normal closure of $\langle Y_k \rangle$ in U_k is $U_k \cap A_k$.*

Proof. Let $a \in \mathbb{N}$ and $(u_{1,k}, \dots, u_{a,k} \mid k \geq 0)$ be a generators sequence with $\langle u_{1,k}, \dots, u_{a,k} \rangle = U_k$. Further, let $(u_1, \dots, u_a)$ be the image of $(u_{1,k}, \dots, u_{a,k})$ under the natural homomorphism $G_k \twoheadrightarrow P$ and let $(v_1, \dots, v_m)$ be an induced polycyclic sequence of $\langle u_1, \dots, u_a \rangle$. Then there exist words $\omega_1, \dots, \omega_m$ and $\omega_1^*, \dots, \omega_a^*$ in the free groups of rank a and m, respectively, such that

$$v_i = \omega_i(u_1, \dots, u_a) \text{ and } u_j = \omega_j^*(v_1, \dots, v_m).$$

Denote $v_{i,k} := \omega(u_{1,k}, \dots, u_{a,k})$ and $Z_k := (v_{1,k}, \dots, v_{m,k})$, and note that Z_k has entries in $U_k = \langle u_{1,k}, \dots, u_{a,k} \rangle$. Evidently, the image of Z_k under the natural homomorphism $G_k \twoheadrightarrow G_k/A_k \cong P$ is an induced polycyclic sequence of U_kA_k/A_k. By Lemma 63 the family $(Z_k \mid k \geq 0)$ is a generators sequence.

Now let us define Y_k. For this purpose, let $R = \Omega(v_1, \dots, v_m)$ be the set of relators of the pc-presentation of $\langle u_1, \dots, u_a \rangle$ with respect to the polycyclic sequence $(v_1, \dots, v_m)$. In particular R is a finite set of words in the free group of rank m. For $1 \leq i \leq a$ put $\tilde{u}_{i,k} := \omega_i^*(v_{1,k}, \ldots, v_{m,k})^{-1} u_{i,k}$, and define $Y_{k,1} := (\tilde{u}_{i,k} \mid 1 \leq i \leq a)$ and $Y_{k,2} := (r(v_{1,k}, \ldots, v_{m,k}) \mid r \in R)$. Further, let Y_k denote the concatenation of $Y_{k,1}$ and $Y_{k,2}$. Note that Y_k has entries in U_k and that $(Y_k \mid k \geq 0)$ is a generators sequence by Lemma 63. It remains to show that the normal closure of $\langle Y_k \rangle$ in U_k is $U_k \cap A_k$. We write $\langle\langle Y_k \rangle\rangle$, $\langle\langle Y_{k,1} \rangle\rangle$ and $\langle\langle Y_{k,2} \rangle\rangle$ to denote the normal closures of Y_k, $Y_{k,1}$ and $Y_{k,2}$ in U_k, respectively. Then

$$\langle Z_k \rangle \cap A_k = \langle v_{1,k}, \ldots, v_{m,k} \rangle \cap A_k = \langle\langle r(v_{1,k}, \ldots, v_{m,k}) \mid r \in R \rangle\rangle = \langle\langle Y_{k,2} \rangle\rangle$$

holds by definition of R. Since $u_i = \omega_i^*(v_1, \ldots, v_m)$, the element $\tilde{u}_{i,k} = \omega_i^*(v_{1,k}, \ldots v_{m,k})^{-1} u_{i,k}$ lies in A_k and as a result $\langle\langle Y_k \rangle\rangle = \langle\langle Y_{k,1} \rangle\rangle \langle\langle Y_{k,2} \rangle\rangle \leq U_k \cap A_k$. Next we show that $U_k \cap A_k \leq \langle\langle Y_k \rangle\rangle$. For this purpose, let u denote an arbitrary element of $U_k \cap A_k$. Note that $\langle \tilde{u}_{i,k}, Z_k \rangle = \langle u_{i,k}, Z_k \rangle$ and hence $\langle\langle Y_{k,1} \rangle\rangle \langle Z_k \rangle = U_k$. Thus there exist elements $v \in \langle\langle Y_{k,1} \rangle\rangle$ and $w \in \langle Z_k \rangle$ with $u = vw$. As $\langle\langle Y_{k,1} \rangle\rangle$ is a subgroup of $U_k \cap A_k$ and $u \in U_k \cap A_k$, the element w lies in $\langle Z_k \rangle \cap A_k = \langle\langle Y_{k,2} \rangle\rangle$. This yields $u \in \langle\langle Y_{k,1} \rangle\rangle \langle\langle Y_{k,2} \rangle\rangle = \langle\langle Y_k \rangle\rangle$ and the result follows. □

Lemma 65. *Let $(U_k \mid k \geq 0)$ be parametrized. Then $(U_k \cap A_k \mid k \geq 0)$ is parametrized.*

Proof. By Lemma 64 there is a generators sequences $(Y_k \mid k \geq 0)$ such that $U_k \cap A_k = \langle y^u \mid y \in \langle Y_k \rangle, u \in U_k \rangle$. Since U_k is parametrized, the group $U_k A_k / A_k$ is naturally isomorphic to $U_{k+1} A_{k+1} / A_{k+1}$ for every $k \geq 0$. Hence there exists a generators sequence $(Q_k \mid k \geq 0)$ with $U_k A_k / A_k = \{z A_k \mid z \in Q_k\}$. As A_k is abelian, we obtain $U_k \cap A_k = \langle y^z \mid y \in Y_k, z \in Q_k \rangle$. Denote $(z_{1,k}, \ldots, z_{a,k}) := Q_k$ and note that $(Y_k^{z_{i,k}} \mid k \geq 0)$ is a generators sequence by Lemma 63. In particular the concatenation of the sequences $Y_k^{z_{i,k}}$ with $1 \leq i \leq a$ also forms a generators sequence. Thus $(U_k \cap A_k \mid k \geq 0)$ is parametrized. □

We need further lemmata for a characterization of parametrized families. An easy observation yields

Lemma 66. *Let $a, m \in \mathbb{N}$ and $C, C' \in \mathbb{Z}_p^{a \times a}$ and $D, D' \in \mathbb{Q}_p^{a \times a}$. Then for $k \gg 0$ the matrix $C + p^k D$ lies in $\mathrm{GL}_a(\mathbb{Z}_p)$ if and only $C \in \mathrm{GL}_a(\mathbb{Z}_p)$. Further, for $k \gg 0$ the product $(C + p^k D) \cdot (C' + p^k D')$ is an element of $\mathbb{Z}_p^{a \times a}$ and*

$$(C + p^k D) \cdot (C' + p^k D') = CC' + p^k (CD' + DC') \mod p^{k+m}.$$

Lemma 67. *Let a, b and m be natural numbers and let A and B be matrices in $\mathbb{Z}_p^{a \times b}$. Then there exist matrices $C \in \mathbb{Z}_p^{a \times a}$, $D \in \mathbb{Q}_p^{a \times a}$ such that for all large k the matrix $C + p^k D$ is invertible in $\mathbb{Z}_p^{a \times a}$ and the product $(C + p^k D)(A + p^k B)$ is in row echelon form modulo p^{k+m}.*

Proof. By Lemma 66 it suffices to show that there exist $o \in \mathbb{N}$ and matrices $C_1, \ldots, C_o \in \mathrm{GL}_a(\mathbb{Z}_p)$ and $D_1, \ldots, D_o \in \mathbb{Z}_p^{a \times a}$ such that $(C_1 + p^k D_1) \cdots (C_o + p^k D_o)(A + p^k B)$ is in row echelon form modulo p^{k+m} for large k. For $a = 1$ the statement is clear, so let us consider $a = 2$. Let α and β denote the first columns of A and B, respectively. We may assume that $\alpha + p^k \beta \neq 0 \bmod p^{k+m}$. If $\alpha = 0$ or $\beta = 0$, then there exists a matrix $C \in \mathrm{GL}_2(\mathbb{Z}_p)$ such that $C(\alpha + p^k \beta) \bmod p^{k+m}$ is in row echelon form. Otherwise there exists a matrix $C \in \mathrm{GL}_2(\mathbb{Z}_p)$ such that $C(\alpha + p^k \beta) = (\alpha_1 + p^k \beta_1, p^k \beta_2)^{tr}$ with $\beta_1, \beta_2 \in \mathbb{Z}_p$ and $0 \neq \alpha_1 \in \mathbb{Z}_p$. Then for

$$F = \begin{pmatrix} 0 & 0 \\ -\alpha_1^{-1} \beta_2 & 0 \end{pmatrix}$$

the matrix $(I_2 + p^k F)C(A + p^k B)$ is in row echelon form modulo p^{k+m} for $k \gg 0$.
So arguing by induction we may assume that the result holds for a. Now let A and B be matrices in $\mathbb{Z}_p^{a+1\times b}$. Without loss of generality the first column of $A + p^k B$ is nontrivial modulo p^{k+m}. By inductive hypothesis there exist square matrices $C_1, C_2 \in \mathrm{GL}_a(\mathbb{Z}_p)$ and $D_1, D_2 \in \mathbb{Q}_p^{a\times a}$ such that for large k only the first entry of the first column of

$$E_k := \mathrm{Diag}(C_1 + p^k D_1, 1) \cdot \mathrm{Diag}(1, C_2 + p^k D_2) \cdot (A + p^k B)$$

is nontrivial modulo p^{k+m}. Further the inductive hypothesis yields the existence of matrices $C_3 \in \mathrm{GL}_a(\mathbb{Z})$ and $D_3 \in \mathbb{Q}_p^{a\times a}$ such that the lower right $a \times b$ submatrix of $F_k := \mathrm{Diag}(1, C_3 + p^k D_3) \cdot E_k$ is in row echelon form modulo p^{k+m} for $k \gg 0$. Since all but the first entry of the first column of F_k is trivial modulo p^{k+m}, the matrix F_k is also in row echelon form modulo p^{k+m} and the result follows. □

Lemma 68. *Let $(U_k \mid k \geq 0)$ be parametrized with $U_k \leq A_k$ and with generators sequence $(v_{1,k}, \ldots, v_{a,k} \mid k \geq 0)$, say. Then there exists a generators sequence $(Z_k \mid k \geq 0)$ such that for large k the sequence Z_k is an induced polycyclic sequence of U_k with respect to X_{A_k}.*
Further, for every generators sequence $(w_k \mid k \geq 0)$ with $w_k \in U_k$ there exist a natural number e and words ω and ω^* in the free group of rank a such that $w_k = \omega(v_{1,k}, \ldots, v_{a,k}) \cdot \omega^*(v_{1,k}, \ldots, v_{a,k})^{p^{k-e}}$ for large k.

Proof. We may assume that there are $c \in \mathbb{N}_0$, $v = (v_1, \ldots, v_a) \in S^a$ and $w = (w_1, \ldots, w_a) \in T^a$ with $\mathcal{E}_c(v, w) = (v_{1,k}, \ldots, v_{a,k} \mid k \geq 0)$. Further, denote

$$A := \begin{pmatrix} \exp_{X_T}(v_1) \\ \vdots \\ \exp_{X_T}(v_a) \end{pmatrix} \in \mathbb{Z}_p^{a\times d} \text{ and } B := \begin{pmatrix} \exp_{X_T}(w_1) \\ \vdots \\ \exp_{X_T}(w_a) \end{pmatrix} \in \mathbb{Z}_p^{a\times d}.$$

The row space of $A + p^{k-c}B$ modulo $p^{k+n'}$ corresponds to U_k. In what follows, we construct induced polycyclic sequences $Z_k = (z_{1,k}, \ldots, z_{b,k})$ of all U_k simultaneously, and we show that there exist a natural number f and words $\omega_1, \ldots, \omega_b, \omega_1^*, \ldots, \omega_b^*$ in the free group of rank a such that

$$z_{i,k} \quad = \quad \omega_i(v_{1,k}, \ldots, v_{a,k}) \cdot \omega_i^*(v_{1,k}, \ldots, v_{a,k})^{p^{k-f}} \tag{6.3.3}$$

for $k \gg 0$. By Lemma 67 there exist a natural number f and matrices $C, D \in \mathbb{Z}_p^{a\times a}$ such that for large k the matrix $C + p^{k-f}D$ is invertible in $\mathbb{Z}_p^{a\times a}$ and the product $F_k := (C + p^{k-f}D)(A + p^{k-c}B)$ is in row echelon form modulo $p^{k+n'}$. Let k be sufficiently large, put $i_* := \min\{j \mid (F_k)_{1,j} \neq 0 \bmod p^{k+n'}\}$ and write ω_1 and ω_1^* to denote the words $\prod_{j=1}^a x_j^{C_{1,j}}$ and $\prod_{j=1}^a x_j^{D_{1,j}}$ in the free group of rank a, respectively. Then $z_{1,k} := \omega_1(v_{1,k}, \ldots, v_{a,k}) \cdot \omega_1^*(v_{1,k}, \ldots, v_{a,k})^{p^{k-f}}$ is the first element of an induced polycyclic sequence of U_k, that is, $\langle z_{1,k}, U_k \cap G_k(q + i_* + 1)\rangle = U_k$. Since $(v_{1,k}, \ldots, v_{a,k} \mid k \geq 0)$ is a generators sequence with entries in the abelian group A_k, it is straightforward to observe that $(z_{1,k} \mid k \geq 0)$ is a generators sequence.
Let $|\cdot|_p$ denote the p-adic absolute value and note that $\mathrm{Diag}(p^{k+n'} \cdot |(F_k)_{1,i_*}|_p, 1, \ldots, 1) \cdot F_k$ corresponds to $U_k \cap G_k(q + i_* + 1)$. Thus we can determine similarly to above words ω_2 and ω_2^* describing a second element of an induced polycyclic sequence. So arguing by induction on d yields the existence of a generators sequence $(z_{1,k}, \ldots, z_{b,k} \mid k \geq 0)$ of induced polycyclic sequences, a natural number f and words $\omega_1, \ldots, \omega_b, \omega_1^*, \ldots, \omega_b^*$ satisfying Equation (6.3.3).

Now let $(w_k \mid k \geq 0)$ denote a generators sequence with $w_k \in U_k$. Since $\langle Z_k \rangle = U_k$, there exists $a_k \in \mathbb{Z}_p$ with $\langle z_{1,k}^{a_k}, z_{2,k}, \ldots, z_{b,k} \rangle = \langle w_k, z_{2,k}, \ldots, z_{b,k} \rangle$. As $(w_k \mid k \geq 0)$ and $(z_{1,k} \mid k \geq 0)$ are generators sequences, there exist $e_1 \in \mathbb{N}_0$ and $n_1, n_2 \in \mathbb{Z}_p$ such that $a_k = n_1 + p^{k-e_1} n_2$ for $k \gg 0$. Further, $w'_k := z_{1,k}^{-a_k} w_k$ lies in $\langle z_{2,k}, \ldots, z_{b,k} \rangle$ and $(w'_k \mid k \geq 0)$ is a generators sequence. Thus arguing by induction on the length of Z_k there exist an integer $\tilde{e}$ and words $\tilde{\omega}_1$, $\tilde{\omega}_2$ in the free group of rank b such that $w_k = \tilde{\omega}_1(z_{1,k}, \ldots, z_{b,k}) \cdot \tilde{\omega}_2(z_{1,k}, \ldots, z_{b,k})^{p^{k-\tilde{e}}}$ for large k. Since A_k is abelian and $z_{i,k}$ equals $\omega_i(v_{1,k}, \ldots, v_{a,k}) \cdot \omega_i^*(v_{1,k}, \ldots, v_{a,k})^{p^{k-f}}$ for $k \gg 0$, the result follows. □

The next lemma deals with induced polycyclic sequences of the product of parametrized families.

Lemma 69. *Let $(U_k \mid k \geq 0)$ and $(V_k \mid k \geq 0)$ be parametrized families with U_k, $V_k \leq A_k$. Then there exist $b \in \mathbb{N}$ and generators sequences $(u_{1,k}, \ldots, u_{b,k} \mid k \geq 0)$ and $(v_{1,k}, \ldots, v_{b,k} \mid k \geq 0)$ with entries in U_k and V_k, respectively, such that the sequence $(w_{1,k}, \ldots, w_{b,k} \mid k \geq 0)$ defined by $w_{i,k} := u_{i,k} \cdot v_{i,k}$ is a generators sequence and $(w_{1,k}, \ldots, w_{b,k})$ is an induced polycyclic sequence of $U_k V_k$ with respect to X_{A_k} for $k \gg 0$.*

Proof. Let $(x_{1,k}, \ldots, x_{a,k})$ and $(y_{1,k}, \ldots, y_{a,k})$ be generators sequences of U_k and V_k, respectively. Since $U_k V_k$ is generated by $x_{1,k}, \ldots, x_{a,k}, y_{1,k}, \ldots, y_{a,k}$, the family $(U_k V_k \mid k \geq 0)$ is parametrized. Then by Lemma 68 there exists a generators sequence $(Z_k \mid k \geq 0) = (z_{1,k}, \ldots, z_{b,k} \mid k \geq 0)$ such that Z_k is an induced polycyclic sequence of $U_k V_k$ for $k \gg 0$. Further, Lemma 68 yields the existence of a natural number e and words $\omega_1, \ldots, \omega_b$ and $\omega_1^*, \ldots, \omega_b^*$ in the free group of rank a such that

$$z_{i,k} = \omega_i(x_{1,k}, \ldots, x_{a,k}, y_{1,k}, \ldots, y_{a,k}) \cdot \omega_i^*(x_{1,k}, \ldots, x_{a,k}, y_{1,k}, \ldots, y_{a,k})^{p^{k-e}}.$$

Since A_k is abelian, there are words $\omega_{i,1}$ and $\omega_{i,2}$ such that the product of $\omega_{i,1}(x_{1,k}, \ldots, x_{a,k})$ and $\omega_{i,2}(y_{1,k}, \ldots, y_{a,k})$ equals $\omega_i(x_{1,k}, \ldots, y_{a,k})$. Similarly, we define $\omega_{i,1}^*$ and $\omega_{i,2}^*$. Now, let $u_{i,k}$ and $v_{i,k}$ denote $\omega_{i,1}(x_{1,k}, \ldots, x_{a,k}) \cdot \omega_{i,1}^*(x_{1,k}, \ldots, x_{a,k})^{p^{k-e}}$ and $\omega_{i,2}(y_{1,k}, \ldots, y_{a,k}) \cdot \omega_{i,2}^*(y_{1,k}, \ldots, y_{a,k})^{p^{k-e}}$, respectively. It is straightforward to observe that $(u_{1,k}, \ldots, u_{b,k} \mid k \geq 0)$ and $(v_{1,k}, \ldots, v_{b,k} \mid k \geq 0)$ are generators sequences. The result follows. □

Lemma 64, 65 and Lemma 68 yield a characterization of parametrized families.

Theorem 70. *Let $\mathcal{U} = (U_k \mid k \geq 0)$ be a family of subgroups $U_k \leq G_k$. Then $\mathcal{U}$ is parametrized if and only if there exists a generators sequence $(v_{1,k}, \ldots, v_{a,k} \mid k \geq 0)$ with $a \in \mathbb{N}$ such that $(v_{1,k}, \ldots, v_{a,k})$ is an induced polycyclic sequence of U_k w.r.t. X_k for all but finitely many k.*

As a consequence of Theorem 70 the sequence of indices $[G_k : U_k]$ induced by a parametrized family $(U_k \mid k \geq 0)$ can be described by a polynomial.

Lemma 71. *Let $(U_k \mid k \geq 0)$ be parametrized. Then there exist $j \in \{0, \ldots, d\}$ and $\alpha \in \mathbb{Q}$ such that $[G_k : U_k] = \alpha p^{jk}$ for large $k \geq 0$.*

Proof. Note that $[G_k : U_k] = [G_k : U_k A_k] \cdot [A_k : U_k \cap A_k]$ and $[G_k : U_k A_k] = [G_0 : U_0 A_0]$. Thus it suffices to consider $[A_k : U_k \cap A_k]$. By Theorem 70 there exist a natural number m and a generators sequence $(f_{1,k}, \ldots, f_{m,k} \mid k \geq 0)$ such that $(f_{1,k}, \ldots, f_{m,k})$ is an induced polycyclic sequence of $(U_k \cap A_k)$ for $k \gg 0$. Thus for $k \gg 0$ the index $[A_k : U_k \cap A_k]$ equals $p^{(n'+k)\cdot(d-m)} \cdot \prod_{i=1}^{m} |\mathrm{lead}_{X_k}(f_{i,k})|_p^{-1}$, where $\mathrm{lead}_{X_k}(f_{i,k})$ and $|\cdot|_p$ denote the leading exponent of $f_{i,k}$ and the p-adic absolute value, respectively. Since $|\mathrm{lead}_{X_k}(f_{i,k})|_p^{-1}$ is either constant or of the form p^{k-c} with $c \in \mathbb{Z}$, the result follows. □

The following proposition is another application of Theorem 70.

Proposition 72. *Let $(U_k \mid k \geq 0)$ be parametrized. Then the family $(U_k' \mid k \geq 0)$ is also parametrized.*

Proof. By Theorem 70 there are a natural number m and a generators sequence $(f_{1,k}, \dots, f_{m,k} \mid k \geq 0)$ such that $(f_{1,k}, \dots, f_{m,k})$ is an induced polycyclic sequence of U_k for $k \gg 0$. Evidently, the group U_k' is the normal closure of $\langle [f_{i,k}^{\pm 1}, f_{j,k}^{\pm 1}] \mid 1 \leq i, j \leq m \rangle$ in U_k. Since $U_k A_k / A_k$ is finite and naturally isomorphic to $U_0 A_0 / A_0$, there exists a generators sequence $(v_{1,k}, \dots, v_{a,k} \mid k \geq 0)$ with $\{v_{1,k}, \dots, v_{a,k}\} = U_k A_k / A_k$ by Lemma 63. As A_k is abelian, the derived subgroup U_k' equals $\langle [f_{i,k}^{\pm 1}, f_{j,k}^{\pm 1}]^{v_{l,k}} \mid 1 \leq i, j \leq m \text{ and } 1 \leq l \leq a \rangle$ and hence U_k' is parametrized by Lemma 63. □

6.4 Intersection of parametrized families

The aim of this section is to show that for two parametrized families $(U_k \mid k \geq 0)$ and $(V_k \mid k \geq 0)$ the family $(U_k \cap V_k \mid k \geq 0)$ is also parametrized. First, we show that this is the case if U_k and V_k are subgroups of A_k.
We say that a matrix $A = (a_{i,j})_{i,j}$ is a diagonal matrix if $a_{i,j} = 0$ for $i \neq j$. Note that diagonal matrices are not necessarily square matrices by our definition.

Lemma 73. *Let a, b and m be natural numbers and let A and B be matrices in $\mathbb{Z}_p^{a \times b}$. Then there are square matrices $C \in \mathrm{GL}_a(\mathbb{Z}_p)$, $D \in \mathbb{Q}_p^{a \times a}$ and $E \in \mathrm{GL}_b(\mathbb{Z}_p)$, $F \in \mathbb{Q}_p^{b \times b}$ and diagonal matrices D_1 and D_2 over $\mathbb{Z}_p$ of suitable dimensions such that for $k \gg 0$*

$$(C + p^k D)(A + p^k B)(E + p^k F) = \mathrm{Diag}(D_1, p^k D_2) \mod p^{k+m}$$

is the Smith normal form of $A + p^k B \mod p^{k+m}$ in $(\mathbb{Z}/p^{k+m}\mathbb{Z})^{a \times b}$.

Proof. For natural numbers r, s we write $0_{r,s}$ to denote the null matrix in $\mathbb{Z}_p^{r \times s}$. Let $m \in \mathbb{N}$, $D_1 \in \mathrm{GL}_m(\mathbb{Z}_p)$, $M \in \mathrm{GL}_a(\mathbb{Z}_p)$ and $N \in \mathrm{GL}_b(\mathbb{Z}_p)$ be such that the matrix $MAB = \mathrm{Diag}(D_1, 0_{a-m,b-m})$ is the Smith normal form of A. Further, let $B_1 \in \mathbb{Z}_p^{m \times m}$, $B_2 \in \mathbb{Z}_p^{m \times b-m}$, $B_3 \in \mathbb{Z}_p^{a-m \times m}$ and $B_4 \in \mathbb{Z}_p^{a-m \times b-m}$ be matrices with

$$S_k := M(A + p^k B)N = \begin{pmatrix} D_1 & 0_{m,b-m} \\ 0_{a-m,m} & 0_{a-m,b-m} \end{pmatrix} + p^k \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

Thus for $k \gg 0$ Lemma 66 yields

$$\begin{pmatrix} I_m - p^k B_1 D_1^{-1} & 0_{m \times a-m} \\ -p^k B_3 D_1^{-1} & I_{a-m} \end{pmatrix} S_k \begin{pmatrix} I_m & -p^k D_1^{-1} B_2 \\ 0_{a-m \times m} & I_{a-m} \end{pmatrix} = \mathrm{Diag}(D_1, p^k B_4) \mod p^{k+c}.$$

Evidently, there exist $K \in \mathrm{GL}_{a-m}(\mathbb{Z}_p)$, $L \in \mathrm{GL}_{b-m}(\mathbb{Z}_p)$ and a diagonal matrix D_2 such that $\mathrm{Diag}(I_m, K) \cdot \mathrm{Diag}(D_1, p^k B_4) \cdot \mathrm{Diag}(I_m, L) = \mathrm{Diag}(D_1, p^k D_2)$ for $k \gg 0$. The result follows by Lemma 66. □

Lemma 74. *Let $(U_k \mid k \geq 0)$ and $(V_k \mid k \geq 0)$ be parametrized with $U_k, V_k \leq A_k$. Then $(U_k \cap V_k \mid k \geq 0)$ is also parametrized.*

Proof. Let $a, b \in \mathbb{N}$ and let $(E_k \mid k \geq 0) = (e_{1,k}, \dots, e_{a,k} \mid k \geq 0)$ and $(F_k \mid k \geq 0) = (f_{1,k}, \dots, f_{b,k} \mid k \geq 0)$ be generators sequences of $(U_k \mid k \geq 0)$ and $(V_k \mid k \geq 0)$, respectively. Further, let M_k denote the exponential matrix of E_k, that is, $M_k = (\exp_{X_{A_k}}(e_{1,k})^{tr}, \dots, \exp_{X_{A_k}}(e_{a,k})^{tr})^{tr}$. Let N_k be the exponential matrix of F_k. By Lemma 73 there are square matrices $C \in \mathrm{GL}_{a+b}(\mathbb{Z}_p)$, $D \in \mathbb{Q}_p^{a+b \times a+b}$, diagonal matrices D_1 and D_2 over $\mathbb{Z}_p$ of suitable dimensions and a natural number c such that for large k the Smith normal form of $(M_k^{tr}, N_k^{tr})^{tr} \mod p^{k+n'}$ in $(\mathbb{Z}/p^{k+n'}\mathbb{Z})^{a+b \times d}$ is $\mathrm{Diag}(D_1, p^{k-c}D_2)$ with corresponding row transformation matrix $(C + p^k D) \mod p^{k+n'}$. Thus there exist a natural number f and square matrices $C', D' \in \mathbb{Z}_p^{a+b \times a+b}$ such that for $k \gg 0$ the row space of $C' + p^{k-f}D' \mod p^{n'+k}$ is the kernel of the linear system $v(M_k^{tr} N_k^{tr})^{tr} = 0 \mod p^{n'+k}$. Let C_k denote the matrix consisting of the first a columns of $C' + p^{k-f}D'$. Then the rows of $C_k \cdot M_k$ correspond to a generating set of $U_k \cap V_k$ and the result follows by Lemma 66. □

Now let $(U_k \mid k \geq 0)$ and $(V_k \mid k \geq 0)$ be parametrized families of subgroups which do not necessarily lie in A_k. We will show that $(U_k \cap V_k \mid k \geq 0)$ is also parametrized. For this purpose, we use the results of [12, Section 8.8.1 Intersections]. First, denote $K_k := U_k \cap V_k A_k$ and $L_k := U_k A_k \cap V_k$. Hence

$$U_k \cap V_k = (U_k \cap V_k A_k) \cap (U_k A_k \cap V_k) = K_k \cap L_k.$$

Lemma 75 ([12, Lemma 8.53])**.** *Let M_k be $L_k \cap A_k = V_k \cap A_k$. Then*

(i) *M_k is normalized by K_k and thus K_k acts on A_k/M_k by conjugation;*

(ii) *Every $g \in K_k$ can be written as $g = v_g t_g$ for $v_g \in V_k$ and $t_g \in A_k$ and the coset $t_g M_k$ is uniquely defined by g;*

(iii) *$\theta_k : K_k \to A_k/M_k,\ g \mapsto t_g M_k$ is a well-defined derivation.*

Lemma 76. *The intersection $U_k \cap V_k$ is $\mathrm{Ker}\, \theta_k$.*

Proof. Let $g \in K_k$ and let $v_g \in V_k$, $t_g \in A_k$ be such that $g = v_g t_g$. Then t_g lies in $M_k = V_k \cap A_k$ if and only if $t_g \in V_k$. This is equivalent to $g \in V_k$ and hence $\mathrm{Ker}\, \theta_k = K_k \cap V_k = U_k \cap V_k$. □

Thus it suffices to show that $(\mathrm{Ker}\, \theta_k \mid k \geq 0)$ is parametrized. In order to do this we need

Lemma 77. *Let $(W_k \mid k \geq 0)$ be a parametrized family. Further, let*

$$\varphi_k : G_k \twoheadrightarrow G_k A_k / A_k \to G_0 A_0 / A_0$$

be the composition of the natural homomorphism and isomorphism. Let W be the subgroup of $G_0 A_0 / A_0$ with $W = \varphi_k(W_k)$ for every k and denote $a := |W|$. Then there exists a generators sequence $(w_{1,k}, \dots, w_{a,k} \mid k \geq 0)$ such that $w_{i,k} \in W_k$ and the image of $\{w_{1,k}, \dots, w_{a,k}\}$ under φ_k equals W.

Proof. Let w be an element of W. By Theorem 70 there is a generators sequence $(z_{1,k}, \dots, z_{b,k} \mid k \geq 0)$ with $b \in \mathbb{N}$ such that $(z_{1,k}, \dots, z_{b,k})$ is an induced polycyclic sequence of W_k for $k \gg 0$. Thus there exists a word ω in the free group of rank b with $\varphi(\omega(z_{1,k}, \dots, z_{b,k})) = w$. The result follows by Lemma 63. □

Now we are able to prove the main result of this subsection.

Theorem 78. *Let $(U_k \mid k \geq 0)$ and $(V_k \mid k \geq 0)$ be parametrized. Then the family $(U_k \cap V_k \mid k \geq 0)$ is also parametrized.*

Proof. Let K_k, L_k and M_k be as in Lemma 75. Further let $(s_{1,k}, \ldots, s_{m,k})$ and $(s'_{1,k}, \ldots, s'_{m',k})$ be generators sequences of U_k and V_k, respectively. By Lemma 76, we have $U_k \cap V_k = \operatorname{Ker} \theta_k$. First, we describe the images of θ_k and then we show that $(\operatorname{Ker} \theta_k \mid k \geq 0)$ is parametrized. We write

$$\begin{aligned}\varphi_k :& U_k \longrightarrow U_k A_k / A_k \longrightarrow U_0 A_0 / A_0 \text{ and} \\ \mu_k :& V_k \longrightarrow V_k A_k / A_k \longrightarrow V_0 A_0 / A_0,\end{aligned}$$

to denote the compositions of natural homomorphisms and isomorphisms. Let W be $(U_0 A_0 \cap V_0 A_0)/A_0$ and let a be the order of $|W|$. Then by Lemma 77 there are generators sequences $(u_{1,k}, \ldots, u_{a,k} \mid k \geq 0)$ and $(v_{1,k}, \ldots, v_{a,k} \mid k \geq 0)$ such that $\varphi_k(u_{i,k}) = \mu_k(v_{i,k})$ and W equals $\{\varphi_k(u_{1,k}), \ldots, \varphi_k(u_{a,k})\}$. Note that the inverse images of W under φ_k and μ_k are K_k and L_k, respectively, and that $\operatorname{Ker} \varphi_k = U_k \cap A_k = K_k \cap A_k$ and $\operatorname{Ker} \mu_k = L_k \cap A_k$. Further, $\varphi_k(v_{i,k}) = \mu_k(u_{i,k})$ yields $v_{i,k} A_k = u_{i,k} A_k$, and hence $\theta_k(u_{i,k}) = \theta_k(v_{i,k} v_{i,k}^{-1} u_{i,k}) = v_{i,k}^{-1} u_{i,k} M_k$. Denote $N_k := \operatorname{Ker} \varphi_k = U_k \cap A_k$ and let $(w_{1,k}, \ldots, w_{a,k} \mid k \geq 0)$ be the generators sequence defined by $w_{i,k} = v_{i,k}^{-1} u_{i,k}$. Now, let k be sufficiently large. For each $g \in K_k$ there are $1 \leq i_g \leq a$ and $s_g \in N_k$ with $g = u_{i_g,k} s_g$, in particular

$$\theta_k(g) = \theta(u_{i_g,k})^{s_g} \cdot \theta_k(s_g) = v_{i_g,k}^{-1} u_{i_g,k} s_g M_k = w_{i_g,k} s_g M_k.$$

Thus for $1 \leq i \leq a$ there exists an element $s \in N_k$ with $\theta_k(u_{i,k} s) = 1$ if and only if $w_{i,k} \in M_k N_k$. Summing up, we obtain

$$\operatorname{Ker} \theta_k = \dot{\bigcup}_{1 \leq i \leq a, w_{i,k} \in M_k N_k} \{u_{i,k} s \mid s \in N_k \text{ with } w_{i,k} s \in M_k\}.$$

In what follows we investigate $\operatorname{Ker}(\theta_k)$ in more detail. By Theorem 70 the families $(M_k \mid k \geq 0) = (V_k \cap A_k \mid k \geq 0)$ and $(N_k \mid k \geq 0) = (U_k \cap A_k \mid k \geq 0)$ are parametrized. Thus for $1 \leq i \leq a$ the family $(\langle w_{i,k}, M_k N_k \rangle \mid k \geq 0)$ is also parametrized. Hence by Lemma 71 there exists a natural number e such that for $1 \leq i \leq a$ and $k \geq e$ the element $w_{i,k}$ lies in $M_k N_k$ if and only if $w_{i,e} \in M_e N_e$.

Now, let i denote a natural number with $1 \leq i \leq a$ and $w_{i,e} \in M_e N_e$. By Lemma 69 there exist $b \in \mathbb{N}$ and generators sequences $(z_{1,1,k}, \ldots, z_{1,b,k} \mid k \geq 0)$ and $(z_{2,1,k}, \ldots, z_{2,b,k} \mid k \geq 0)$ with entries in M_k and N_k, respectively, such that the sequence $(z_{1,k}, \ldots, z_{b,k} \mid k \geq 0)$ defined by $z_{j,k} := z_{1,j,k} \cdot z_{2,j,k}$ is a generators sequence and $(z_{1,k}, \ldots, z_{b,k})$ is an induced polycyclic sequence of $M_k N_k \leq A_k$ for $k \gg 0$. Assume that k is sufficiently large in what follows. By Lemma 68 there are an integer f and words ω, ω^* in the free group of rank b with $w_{i,k} = \omega(z_{1,k}, \ldots, z_{b,k}) \cdot \omega^*(z_{1,k}, \ldots, z_{b,k})^{p^{k-f}}$. Since A_k is abelian, we have $\omega'(z_{1,k}, \ldots, z_{b,k}) = \omega'(z_{1,1,k}, \ldots, z_{1,b,k}) \cdot \omega'(z_{2,1,k}, \ldots, z_{2,b,k})$ for $\omega' \in \{\omega, \omega^*\}$. This yields the existence of a generators sequence with entries $s_{i,k} \in N_k$ satisfying $w_{i,k} s_{i,k} \in M_k$. It follows that for $s \in N_k$ the element $w_{i,k} s = w_{i,k} s_{i,k} s_{i,k}^{-1} s$ lies in M_k if and only if $s_{i,k}^{-1} s \in M_k \cap N_k$, in particular

$$\operatorname{Ker} \theta_k = \dot{\bigcup}_{1 \leq i \leq a, w_{i,e} \in M_e N_e} \{u_{i,k} s_{i,k} t \mid t \in M_k \cap N_k\}.$$

As $(M_k \cap N_k \mid k \geq 0) = (U_k \cap V_k \cap A_k \mid k \geq 0)$ is parametrized by Lemma 74, the family $(\operatorname{Ker} \theta_k \mid k \geq 0)$ is also parametrized and the result follows. $\square$

Chapter 7

Irreducible characters of finite p-groups

Let G and l denote a finite p-group and a natural number, respectively, and recall that we write $\mathrm{N}_l(G)$ to denote the number of irreducible characters $\chi \in \mathrm{Irr}\, G$ with $\chi(1) = p^l$. In this section we describe a method to determine $\mathrm{N}_l(G)$. We say that an irreducible character χ is linearly induced from a subgroup U if there exists a linear character of U which induces to χ. Since finite p-groups are monomial, every irreducible character of G of degree p^l is *linearly induced* from some subgroup of index p^l in G. For a subgroup $U \leq G$ we denote the class of linear characters of U by $\mathrm{Lin}\, U$, and for $\mu \in \mathrm{Lin}\, U$ we write $\mu\!\uparrow_U^G$ to denote the character of G induced by μ. In the following we introduce criteria to decide if a given linearly induced character is irreducible and if two linearly induced characters are equal.

7.1 Deciding irreducibility

Let U be a subgroup of G. For a linear character of U, say χ, we define

$$c_\chi : U\backslash G/U \to \{0,1\}, \quad UgU \mapsto \begin{cases} 1, & \text{if } [g,U] \cap U \subseteq \mathrm{Ker}\, \chi, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 79. *The map c_χ is well-defined.*

Proof. Let $g \in G$ be such that $[g,U] \cap U \subseteq \mathrm{Ker}\, \chi$ and let $u, v \in U$ be arbitrary elements of U. Then for $w \in U$ with $[ugv, w] \in U$ we have $[ugv, w] = ([g, w^u] \cdot [u, w])^v \cdot [v, w] \in U$. This yields $[g, w^u] \in U$ and hence $[g, w^u] \in \mathrm{Ker}\, \chi$. Since U' is a subgroup of $\mathrm{Ker}\, \chi$ and $\mathrm{Ker}\, \chi$ is normal in U, it follows that $[ugv, w] \in \mathrm{Ker}\, \chi$ and thus $[ugv, U] \cap U \subseteq \mathrm{Ker}\, \chi$. $\square$

Further, we define

$$c_U : U\backslash G/U \to \{0,1\}, \quad UgU \mapsto \begin{cases} 1, & \text{if } UgU = U, \\ 0, & \text{otherwise.} \end{cases}$$

Based on this notation we are able to decide irreducibility for linearly induced characters:

Lemma 80. *Let G be a finite p-group, $U \leq G$ and let χ be a linear character of U. Then $\chi\!\uparrow_U^G$ is irreducible if and only if $c_\chi = c_U$.*

Proof. This follows from [24, Theorem 1.3]. □

7.2 Deciding equality of linearly induced characters

Let U and V be different subgroups of the same index in G. We consider a linear character χ of U. Our aim is to decide whether $\chi\!\uparrow_U^G$ is also linearly induced from V and how many different linear characters of U induce to $\chi\!\uparrow_U^G$. For $W \leq G$ we define

$$a_{\chi,W} : W\backslash G/U \to \{0,1\}, \quad WgU \mapsto \begin{cases} 1, & \text{if } (W^g)' \cap U \subseteq \operatorname{Ker}\chi, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\operatorname{Ker}\chi$ is a normal subgroup of U and thus the map $a_{\chi,W}$ is well-defined. The following Lemma is a preparation for the main result of this subsection.

Lemma 81. *Let $g \in G$ and φ be a linear character of U. Then for $W \leq G$ the restriction $\varphi\!\downarrow_{U\cap W^g}$ is extendible to W^g if and only if $a_{\chi,W}(WgU) = 1$, that is, $(W^g)' \cap U$ is a subgroup of $\operatorname{Ker}\varphi$. In this case there are exactly $[W^g : (U \cap W^g)(W^g)']$ extensions.*

Proof. By [16, Corollary (5.5)] the first claim follows. Assume that $(W^g)' \cap U \leq \operatorname{Ker}\varphi$. If λ and μ are extensions of $\varphi\!\downarrow_{U\cap W^g}$, then $\lambda\!\downarrow_{U\cap W^g} \cdot \mu\!\downarrow_{U\cap W^g}^{-1} = \mathrm{id}_{U\cap W^g}$. Thus the number of extensions is $|\operatorname{Irr}(W^g/(U \cap W^g)(W^g)')| = [W^g : (U \cap W^g)(W^g)']$. □

Motivated by Lemma 81, we define

$$a_U : U\backslash G/U \to \mathbb{N}, \quad UgU \mapsto [U^g : (U \cap U^g)(U^g)'].$$

Further, the product $a * a'$ of two double coset maps $a, a' : U\backslash G/U \to \mathbb{N}$ is defined as the sum $\sum_{UgU \in U\backslash G/U} a(UgU) \cdot a'(UgU)$.

Theorem 82. *Let G be a finite p-group and U and V be different subgroups of the same index in G. Further, let χ be a linear character of U such that $\chi\!\uparrow_U^G$ is irreducible. Then*

(i) *the character $\chi\!\uparrow_U^G$ is linearly induced from V if and only if $a_{\chi,V} \neq 0$,*

(ii) *there are exactly $a_{\chi,U} * a_U$ different linear characters of U inducing to $\chi\!\uparrow_U^G$.*

Proof. Let φ be a linear character of V. The Frobenius Reciprocity and Mackey's Theorem yield $(\varphi\!\uparrow_V^G, \chi\!\uparrow_U^G) = \sum_{f \in \operatorname{Rep}(V\backslash G/U)} (\varphi^f\!\downarrow_{V^f\cap U}, \chi\!\downarrow_{V^f\cap U})$, where $\operatorname{Rep}(V\backslash G/U)$ denotes a set of double coset representatives of $V\backslash G/U$. By equality of degrees and irreducibility of $\chi\!\uparrow_U^G$, the character $\chi\!\uparrow_U^G$ is equal to $\varphi\!\uparrow_V^G$ if and only if there is an element $f \in \operatorname{Rep}(V\backslash G/U)$ with $\varphi^f\!\downarrow_{V^f\cap U} = \chi\!\downarrow_{V^f\cap U}$. This f would be unique in $\operatorname{Rep}(V\backslash G/U)$.
Hence $\chi\!\uparrow_U^G$ is not linearly induced from V if and only if $\chi\!\downarrow_{V^f\cap U}$ is not extendible to V^f for all $f \in \operatorname{Rep}(V\backslash G/U)$. By Lemma 81, this is equivalent to $a_{\chi,V}(VfU) = 0$ for all $VfU \in V\backslash G/U$. Thus (i) follows.

It remains to show (ii). Similarly to (i), we obtain the following partition

$$\begin{aligned}
&\{\varphi \in \operatorname{Lin} U \mid \varphi\uparrow_U^G = \chi\uparrow_U^G\} \\
&= \dot{\bigcup}_{f\in \operatorname{Rep}(U\backslash G/U)} \{\varphi \in \operatorname{Lin} U \mid \varphi^f \in \operatorname{Lin}(U^f) \text{ is an extension of } \chi\downarrow_{U^f\cap U}\} \\
&= \dot{\bigcup}_{f\in \operatorname{Rep}(U\backslash G/U),\ a_{\chi,U}(UfU)=1} \{\varphi \in \operatorname{Lin} U \mid \varphi^f \in \operatorname{Lin}(U^f) \text{ is an extension of } \chi\downarrow_{U^f\cap U}\},
\end{aligned}$$

where the last equation follows by Lemma 81. Further, by Lemma 81 there are exactly $[U^f : (U\cap U^f)(U^f)']$ extensions of $\chi\downarrow_{U^f\cap U}$ in $\operatorname{Lin}(U^f)$ for $f \in \operatorname{Rep}(U\backslash G/U)$ with $a_{\chi,U}(UfU) = 1$. The result follows. □

7.3 Determining $\mathrm{N}_l(G)$

The results from the previous subsections yield a formula for $\mathrm{N}_l(G)$ which we exhibit in this section. First we introduce some notation. Let $s \in \mathbb{N}$ and $\mathcal{U} = \{U_1, \ldots, U_s\}$ be a set of different subgroups of index p^l in G such that every irreducible character of degree p^l of G is linearly induced from at least one of these subgroups. As conjugate subgroups induce the same characters, we may assume that the subgroups $U_1, \ldots, U_s$ are pairwise non-conjugate.
For $1 \le j, i \le s$ we define $\mathrm{DCM}(U_j, U_i) := \{a : U_j\backslash G/U_i \to \{0, 1\}\}$ the set of *double coset maps*. Let $\mathrm{DCM}^*(U_j, U_i)$ be the set of maps a in $\mathrm{DCM}(U_j, U_i)$ with $a(U_j 1 U_i) = 1$. For $a \in \mathrm{DCM}^*(U_i, U_i)$ define

$$\begin{aligned}
\mathcal{F}(\mathcal{U}, i, a) &:= \{\chi \in \operatorname{Lin} U_i \mid a_{\chi,U_i} = a, c_\chi = c_{U_i} \text{ and } a_{\chi,U_j} = 0 \text{ for } j < i\} \text{ and} \\
\mathcal{G}(\mathcal{U}, i) &:= \bigcup_{a\in \mathrm{DCM}^*(U_i,U_i)} \{\chi\uparrow_{U_i}^G \mid \chi \in \mathcal{F}(\mathcal{U}, i, a)\}.
\end{aligned}$$

Theorem 83.

(i) $\{\chi \in \operatorname{Irr} G \mid \chi(1) = p^l\} = \dot{\bigcup}_{1\le i\le s} \mathcal{G}(\mathcal{U}, i)$.

(ii) *For $1 \le i \le s$ the set $\mathcal{G}(\mathcal{U}, i)$ has $\sum_{a\in\mathrm{DCM}^*(U_i,U_i)} \frac{1}{a * a_{U_i}} |\mathcal{F}(\mathcal{U}, i, a)|$ elements.*

(iii) $\mathrm{N}_l(G) = \sum_{i=1}^s \sum_{a\in\mathrm{DCM}^*(U_i,U_i)} \frac{1}{a*a_{U_i}} |\mathcal{F}(\mathcal{U}, i, a)|$.

Proof. Let i denote a natural number with $1 \le i \le s$. Lemma 80 and Theorem 82 yield that $\mathcal{G}(\mathcal{U}, i)$ is the set of all irreducible characters of G of degree p^l which are linearly induced from U_i but not from any other group U_j with $j < i$. As all irreducible characters of G of degree p^l are induced from subgroups in $\mathcal{U}$, the result for (i) follows. The second result follows by Theorem 82 (ii) and the third result is a direct consequence of (i) and (ii). □

Part II

Character degrees of finite p-groups by coclass

Chapter 8

Character degrees of a coclass family

This chapter deals with the character degrees of groups in a coclass family. The most familiar examples for coclass families are $(D_{2^{k+3}} \mid k \geq 0)$, $(Q_{2^{k+3}} \mid k \geq 0)$ and $(SD_{2^{k+4}} \mid k \geq 0)$, where D_n, Q_n and SD_n denote the dihedral group, quaternion group and semi-dihedral group of order n, respectively, see Figure 5.1. The irreducible characters of the groups within these families are well-understood, for instance the chacter tables of D_{2^k} can be summarized by a single, parametrized character table, in which only simple terms depending on k occur. Of course, the question arises whether for every coclass family there exists such a parametrized character table. It seems difficult to us to solve this question: one major problem is to describe the conjugacy classes in a parametrized form. We focus on the character degrees.

Let us discuss the character degrees of the families $(D_{2^{k+3}} \mid k \geq 0)$, $(Q_{2^{k+3}} \mid k \geq 0)$ and $(SD_{2^{k+4}} \mid k \geq 0)$. For this purpose, let G denote a group in one of these families. Then G/G' is isomorphic to the Klein four-group. Further, there exists a normal, abelian subgroup of G of index 2. Hence there are exactly 4 linear characters of G and by a result of Seitz [13, Lemma 24.8] the irreducible characters of G have degree at most 2, in particular $\mathrm{N}_l(G) = 0$ for $l \geq 2$. Since $|G| = \sum_{\chi \in \mathrm{Irr}\, G} \chi(1)^2 = \sum_{l \in \mathbb{N}_0} 2^{2l} \cdot \mathrm{N}_l(G)$ we have $\mathrm{N}_1(G) = (|G| - 4)/4$. This yields

$$\begin{array}{l}
\mathrm{N}_0(D_{2^{k+4}}) = \mathrm{N}_0(SD_{2^{k+4}}) = \mathrm{N}_0(Q_{2^{k+4}}) = 4, \\
\mathrm{N}_1(D_{2^{k+4}}) = \mathrm{N}_1(SD_{2^{k+4}}) = \mathrm{N}_1(Q_{2^{k+4}}) = 2^{k+2} - 1 \text{ and} \\
\mathrm{N}_l(D_{2^{k+4}}) = \mathrm{N}_l(SD_{2^{k+4}}) = \mathrm{N}_l(Q_{2^{k+4}}) = 0 \text{ for } l \geq 2.
\end{array}$$

It will turn out that a similar result holds for every coclass family.

Theorem 84. *Let $(G_k \mid k \geq 0)$ be a coclass family and d denote the dimension of the associated pro-p-group.*

- *There exists an integer b such that every irreducible character for every G_k has degree at most p^b.*
- *For every nonnegative integer $0 \leq l \leq b$ there exist a polynomial $f_l(x) \in \mathbb{Q}[x]$ with $deg(f_l) \leq d$ and a natural number w such that $\mathrm{N}_l(G_k) = f_l(p^k)$ for every $k \geq w$.*

In the remainder of this chapter we give a proof of Theorem 84.

Proof of the main theorem

Let $\mathcal{F} = \mathcal{F}(\gamma, \delta, n) = (G_k \mid k \geq 0)$ be a coclass family, and let P, T and $S = E(\gamma)$ be the associated point group, translation subgroup and pro-p-group, respectively. By Lemma 40 we may assume that n is a multiple of d. Further let $T = T_0 \geq T_1 \geq T_2 \geq \cdots$ be the lower P-central series of T and write A_k to denote the factor group T/T_{n+kd}. To ease the notation, we also write T_i and A_k to denote images of the natural embeddings $T_i \hookrightarrow S$, $t \mapsto (1,t)$ and $A_k \hookrightarrow G_k = E(\gamma_{A_k} + p^k \delta_{A_k})$, $t + T_{n+kd} \mapsto (1, t + T_{n+kd})$. In particular we regard T_i and A_k as normal subgroups of S and G_k, respectively. Let l be a natural number. As A_k is an abelian normal subgroup of G_k, the following lemma is a direct consequence of a result by Seitz [13, Lemma 24.8].

Lemma 85. *For $k \in \mathbb{N}_0$ all irreducible characters of G_k are linearly induced from overgroups of A_k.*

Let $\mathcal{U} = \{U_1, \ldots, U_s\}$ be a set of representatives of the conjugacy classes of subgroups of S having T as subgroup and index p^l in S. We write $\mathcal{U}_k = \{U_{1,k}, \ldots, U_{s,k}\}$ to denote the image of $\mathcal{U}$ under the natural surjection $\eta_k : S \twoheadrightarrow G_k : (g,t) \mapsto (g, t + T_{n+kd})$. Recall that S is the group extension of T by P defined by γ whereas the group extension G_k of A_k by P is defined by $\gamma_{A_k} + p^k \delta_{A_k}$. Thus G_k is not a quotient of S and η_k is in general not a group homomorphism. But the images $U_{1,k}$, …, $U_{s,k}$ are groups since $T \leq U_i$. Applying η_k on representatives of double cosets, we see that η_k induces a bijection between $U_j \backslash S / U_i$ and $U_{j,k} \backslash G_k / U_{i,k}$ for $1 \leq j, i \leq s$. Further, η_k induces a bijection between the double coset maps, introduced in Section 7.3, as follows:

$$\mathrm{DCM}(U_j, U_i) \to \mathrm{DCM}(U_{j,k}, U_{i,k}),\ f \mapsto (U_{j,k} g U_{i,k} \mapsto f(U_j \eta_k^{-1}(g) U_i)),$$

where $\eta_k^{-1}(g)$ denotes a preimage of $g \in G_k$ under the surjection $\eta_k : S \to G_k$. By abuse of notation, we also write $\eta_k(f)$ to denote the double coset map induced by η_k for $f \in \mathrm{DCM}(U_j, U_i)$. Recall that for $\chi \in \mathrm{Lin}\, U_{i,k}$ we write $a_{\chi, U_{j,k}} \in \mathrm{DCM}(U_{j,k}, U_{i,k})$ and $c_\chi \in \mathrm{DCM}(U_{i,k}, U_{i,k})$ to denote the coset maps whose supports consist of the double cosets $U_{j,k} g U_{i,k}$ with $(U_{j,k}^g)' \cap U_{i,k} \subseteq \mathrm{Ker}\, \chi$ and of the double cosets $U_{i,k} g U_{i,k}$ with $[g, U_{i,k}] \cap U_{i,k} \subseteq \mathrm{Ker}\, \chi$, respectively, see Chapter 7. Further let $\succ$ denote the product order on the class of integer-valued maps, that is, for two integer-valued maps f and g with the same domain we write $f \succ g$ if and only if $f - g$ has only nonnegative values and $f \neq g$. For $c \in \mathrm{DCM}(U_i, U_i)$ and $a = (a_1, \ldots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$ we define

$$\begin{aligned} \mathcal{F}(\mathcal{U}_k, i, a, c) &:= \{\chi \in \mathrm{Lin}\, U_{i,k} \mid a_{\chi, U_{j,k}} = \eta_k(a_j), c_\chi = \eta_k(c) \text{ for } 1 \leq j \leq i\} \text{ and} \\ \mathcal{H}(\mathcal{U}_k, i, a, c) &:= \{\chi \in \mathrm{Lin}\, U_{i,k} \mid a_{\chi, U_{j,k}} \succeq \eta_k(a_j), c_\chi \succeq \eta_k(c) \text{ for } 1 \leq j \leq i\}. \end{aligned}$$

By definitions of $\mathcal{F}(\mathcal{U}_k, i, a, c)$ and $\mathcal{H}(\mathcal{U}_k, i, a, c)$ it follows directly

Lemma 86. *For $1 \leq i \leq s$, $k \in \mathbb{N}_0$, $c \in \mathrm{DCM}(U_i, U_i)$ and $a = (a_1, \ldots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$ we obtain that*

$$\mathcal{F}(\mathcal{U}_k, i, a, c) = \mathcal{H}(\mathcal{U}_k, i, a, c) \setminus \dot{\bigcup}_{(a',c') \succ (a,c)} \mathcal{F}(\mathcal{U}_k, i, a', c'),$$

where $\succ$ denotes the product order on tuples of double coset maps.

For a map $f \in \mathrm{DCM}(U_j, U_i)$, let $\mathrm{supp}(f) \subseteq S$ be a set of representatives of the support of f. In particular the set $\{U_j g U_i \mid g \in \mathrm{supp}(f)\}$ consists of the double cosets $U_j g U_i$ with $f(U_j g U_i) = 1$. Further, for $c \in \mathrm{DCM}(U_i, U_i)$ and $a = (a_1, \ldots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$ define

$$V_k(a, c) := \bigcap_{\chi \in \mathcal{H}(\mathcal{U}_k, i, a, c)} \mathrm{Ker}\, \chi.$$

Lemma 87. *Let $k \in \mathbb{N}$, $1 \le i \le s$, $c \in \mathrm{DCM}(U_i, U_i)$ and $a = (a_1, \dots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$. Then*

(i) $|\mathcal{H}(\mathcal{U}_k, i, a, c)| = [U_{i,k} : V_k(a, c)]$ *and*

(ii) *the group $V_k(a, c)$ is generated by $U'_{i,k}$, $(U_{j,k}^{\eta_k(g)})' \cap U_{i,k}$ with $1 \le j \le i$ and $g \in supp(a_j)$, and $[\eta_k(h), U_{i,k}] \cap U_{i,k}$ with $h \in supp(c)$.*

Proof. Denote $V := V_k(a, c)$ and $\mathcal{H} := \mathcal{H}(\mathcal{U}_k, i, a, c)$. Let W be the group generated by the groups given in (ii). Note that a linear character $\chi \in \mathrm{Lin}\, U_{i,k}$ lies in $\mathcal{H}$ if and only if $a_{\chi, U_{j,k}}(U_{j,k} g U_{i,k}) = 1$ and $c_\chi(U_{i,k} h U_{i,k}) = 1$ for $1 \le j \le i$, $g \in \eta_k(\mathrm{supp}(a_j))$ and $h \in \eta_k(\mathrm{supp}(c))$. Since the equalities $a_{\chi, U_{j,k}}(U_{j,k} g U_{i,k}) = 1$ and $c_\chi(U_{i,k} h U_{i,k}) = 1$ are equivalent to $(U_{j,k}^g)' \cap U_{i,k} \subseteq \mathrm{Ker}\, \chi$ and $[h, U_{i,k}] \cap U_{i,k} \subseteq \mathrm{Ker}\, \chi$, it follows that $\mathcal{H} = \{\chi \in \mathrm{Lin}\, U_{i,k} \mid W \subseteq \mathrm{Ker}\, \chi\}$. As $U'_{i,k}$ is contained in W, the group $V = \bigcap_{\chi \in \mathcal{H}(\mathcal{U}_k, i, a, c)} \mathrm{Ker}\, \chi$ equals W and $\mathcal{H}$ is naturally isomorphic to $U_{i,k}/V$. The result follows. $\square$

The major difficulty in proving Theorem 84 is to show

Theorem 88. *Let $1 \le i \le s$, $c \in \mathrm{DCM}(U_i, U_i)$ and $a = (a_1, \dots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$. Then there are natural numbers $w(a, c)$, $j = j(a, c)$ with $0 \le j \le d$ and $\alpha = \alpha(a, c) \in \mathbb{Q}$ such that $[U_{i,k} : V_k(a, c)] = \alpha p^{jk}$ for $k \ge w(a, c)$.*

We prove Theorem 88 for mainline groups in Section 8.1 and for off-mainline groups in Section 8.2. Theorem 88 yields

Theorem 89. *There exists a natural number w such that for any $1 \le i \le s$, $c \in \mathrm{DCM}(U_i, U_i)$ and $a = (a_1, \dots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$ there are polynomials $f_{i,a,c}$, $g_{i,a,c} \in \mathbb{Q}[x]$ of degree at most d such that for $k \ge w$*

(i) $|\mathcal{H}(\mathcal{U}_k, i, a, c)| = f_{i,a,c}(p^k)$ *and*

(ii) $|\mathcal{F}(\mathcal{U}_k, i, a, c)| = g_{i,a,c}(p^k)$.

Proof. ad (i): As the set $\mathrm{DCM}(U_j, U_i)$ is finite for $1 \le j \le i \le s$, the result for (i) follows by Lemma 87 and Theorem 88.
ad (ii): Let w be such that (i) holds for $k \ge w$ and assume that k is at least w. If (a, c) is maximal according to the partial order $\succeq$, then we have $|\mathcal{F}(\mathcal{U}_k, i, a, c)| = |\mathcal{H}(\mathcal{U}_k, i, a, c)| = f_{i,a,c}(p^k)$. Thus arguing by induction we may assume that the result holds for $(a', c') \succ (a, c)$. Lemma 86 yields that $|\mathcal{F}(\mathcal{U}_k, i, a, c)|$ equals $f_{i,a,c}(p^k) - \sum_{(a',c') \succ (a,c)} g_{i,a',c'}(p^k)$. $\square$

Proof of Theorem 84. First, note that by Lemma 85 every irreducible character of degree p^l of G_k is linearly induced from at least one of the subgroups in $\mathcal{U}_k = \{U_{1,k}, \dots, U_{s,k}\}$. Further, let $a_{U_{i,k}} : U_{i,k} \backslash G_k / U_{i,k} \to \mathbb{N}$ and $c_{U_{i,k}} : U_{i,k} \backslash G_k / U_{i,k} \to \{0, 1\}$ be defined as in Chapter 7, that is, $a_{U_{i,k}}(U_{i,k} g U_{i,k}) = [U_{i,k}^g : (U_{i,k} \cap U_{i,k}^g)(U_{i,k}^g)']$ and $c_{U_{i,k}}(U_{i,k} h U_{i,k}) = 1$ iff $U_{i,k} h U_{i,k} = U_{i,k}$. Evidently, we have $a_{U_{i,k}} = \eta_k(a_{U_i})$ and $c_{U_{i,k}} = \eta_k(c_{U_i})$. For a double coset map $f \in \mathrm{DCM}^*(U_{i,k}, U_{i,k})$ let further $\mathcal{F}(\mathcal{U}_k, i, f)$ be as in Theorem 83. Since η_k induces a bijection from $\mathrm{DCM}^*(U_i, U_i)$ to $\mathrm{DCM}^*(U_{i,k}, U_{i,k})$, it follows by Theorem 83 that

$$\mathrm{N}_l(G_k) = \sum_{i=1}^{s} \sum_{f \in \mathrm{DCM}^*(U_i, U_i)} |\mathcal{F}(\mathcal{U}_k, i, \eta_k(f))| / (\eta_k(f) * \eta_k(a_{U_i})).$$

For $f \in \mathrm{DCM}^*(U_i, U_i)$ let $a_f = (a_1, \ldots, a_i)$ denote the tuple of coset maps with $a_j = 0 \in \mathrm{DCM}(U_j, U_i)$ for $j < i$, and $a_i = f$. Note that $\mathcal{F}(\mathcal{U}_k, i, \eta_k(f)) = \mathcal{F}(\mathcal{U}_k, i, a_f, c_{U_i})$ and $\eta_k(f) * \eta_k(a_{U_i}) = f * a_{U_i}$. Thus Theorem 84 follows by Theorem 89. □

8.1 Proof of Theorem 88 for mainline groups

Recall that we regard T_i also as normal subgroup of S for every nonnegative i. In what follows we prove Theorem 88 for the case in which all but finitely many groups of the coclass family $\mathcal{F}$ are mainline groups. In the following Section 8.2 we give a proof of Theorem 88 for the general case making this subsection redundant. However the results of Section 8.2 heavily rely on results of Chapter 6 whereas we need for the proof for mainline groups only the following elementary lemmata.

Lemma 90. *If all but finitely many groups of the coclass family $\mathcal{F} = \mathcal{F}(\gamma, \delta, n)$ are isomorphic to mainline groups, then $\mathcal{F}$ is isomorphic to $\mathcal{F}(\gamma, 0, n)$.*

Proof. First note that for every $m \in \mathbb{N}_0$ the factor group S/T_m is isomorphic to a quotient of G_k for some k. By assumption, for every $k \gg 0$ the group G_k is isomorphic to a quotient of a pro-p-group of coclass $\operatorname{cc} G_k = \operatorname{cc} S$. Since there exists only finitely many pro-p-groups of fixed coclass by Theorem 30, it follows that G_k is isomorphic to a quotient of S for $k \gg 0$. In particular we have $G_k \cong E(\gamma_{A_k})$. The result follows. □

Lemma 91. *Let $1 \leq i \leq s$ and $V \leq U_i$ be such that $U_i' \leq V$. Then there are natural numbers $w(i, V)$, $j = j(i, v)$ with $0 \leq j \leq d$ and $\alpha = \alpha(i, V) \in \mathbb{N}$ such that $[U_i/T_{kd} : VT_{kd}/T_{kd}] = \alpha p^{jk}$ for all $k \geq w(i, V)$.*

Proof. First, recall that T is a free $\mathbb{Z}_p$-module of rank d and that $T_{kd} = p^k T$. In particular we have $[T : T_{kd}] = p^{kd}$. The index $[U_i/T_{kd} : VT_{kd}/T_{kd}]$ equals $[U_i : VT] \cdot [T : T_{kd}] \cdot [V \cap T : V \cap T_{kd}]^{-1}$. Thus it remains to show that $[V \cap T : V \cap T_{kd}] = \beta p^{jk}$ for some $\beta \in \mathbb{Q}$, $1 \leq j \leq d$ and for every $k \gg 0$. Evidently, there exists a matrix $A \in \mathbb{Z}_p^{d \times d}$ such that the row space of A corresponds to the intersection $V \cap T$. Let $\mathrm{Diag}(a_1, \ldots, a_c, 0, \ldots, 0) \in \mathbb{Z}_p^{d \times d}$ be the Smith normal form of A with $a_m \neq 0$ and denote $w(i, V) := \max_{1 \leq m \leq c} \log_p(a_m)$. Then for $k \geq w(i, V)$ we have $[V \cap T : V \cap T_{kd}] = \prod_{m=1}^{c} (p^k/a_m) = p^{kc} \prod_{m=1}^{c} (1/a_m)$. The result follows. □

Proof of Theorem 88 for mainline groups. By Lemma 90 we may assume that $\mathcal{F} = \mathcal{F}(\gamma, 0, n)$. Then $G_k = E(\gamma_{A_k})$ is naturally isomorphic to S/T_{n+kd} and the map η_k is the natural homomorphism $S \twoheadrightarrow G_k$ and hence a group homomorphism. It follows that $U_{i,k}' = \eta_k(U_i')$, $(U_{j,k}^{\eta_k(g)})' \cap U_{i,k} = \eta_k((U_j^g)' \cap U_i)$ and $[\eta_k(h), U_{i,k}] \cap U_{i,k} = \eta_k([h, U_i] \cap U_i)$ for g, $h \in S$. Thus by Lemma 87 (ii) there exists a group V such that $U_i' \leq V \leq U_i$ and $\eta_k(V) = V_k(a, c)$. Since n is a multiple of d by assumption and $\eta_k(V)$ is isomorphic to V/T_{n+kd}, the result follows by Lemma 91. □

8.2 Proof of Theorem 88 for off-mainline groups

The results of Chapter 6 enable us to prove Theorem 88 for the general case. First, recall that $U_1, \ldots, U_s$ are subgroups of S with $T \leq U_i$ and that we write $U_{i,k}$ to denote the image of U_i under the natural surjection $\eta_k : S \twoheadrightarrow G_k$, $(g, t) \mapsto (g, t + T_{k+nd})$ for $1 \leq i \leq s$. It follows that $(U_{i,k} \mid k \geq 0)$

is a parametrized family by construction. Further for a double coset map $c \in \mathrm{DCM}(U_i, U_i)$ and a list of double coset maps $a = (a_1, \dots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$ Lemma 87 (ii) yields

$$V_k(a,c) = \langle U'_{i,k}, (U_{j,k}^{\eta_k(g)})' \cap U_{i,k}, [\eta_k(h), U_{i,k}] \cap U_{i,k} \mid 1 \leq j \leq i, g \in \mathrm{supp}(a_j), h \in \mathrm{supp}(c)\rangle.$$

Lemma 92. *Let $1 \leq i \leq s$, $c \in \mathrm{DCM}(U_i, U_i)$ and $a = (a_1, \dots, a_i)$ with $a_j \in \mathrm{DCM}(U_j, U_i)$. Then $(V_k(a,c) \mid k \geq 0)$ is parametrized.*

Proof. Let g be an element of $S = E(\gamma)$. Then $(\eta_k(g) \mid k \geq 0)$ is a generators sequence by definition. Since $(U_{j,k} \mid k \geq 0)$ is parametrized, Lemma 61 yields that $(U_{j,k}^{\eta_k(g)} \mid k \geq 0)$ is also parametrized. Next, we show that $([\eta_k(g), U_{i,k}] \mid k \geq 0)$ is parametrized. For this purpose, let $\{u_1, \dots, u_b\}$ be a set of representatives of U_i/T and let $(x_{q+1,k}, \dots, x_{q+d,k} \mid k \geq 0)$ be a generators sequence of $(A_k \mid k \geq 0)$, where A_k is regarded as subgroup of G_k. As $[h, t \cdot u] = [h,u] \cdot [h,t]^u$ holds for h, t and $u \in G_k$, and A_k is abelian, it follows that

$$\begin{aligned} [\eta_k(g), A_k] &= \langle [\eta_k(g), x_{q+1,k}], \dots, [\eta_k(g), x_{q+d,k}]\rangle \text{ and} \\ [\eta_k(g), U_{i,k}] &= \langle [\eta_k(g), \eta_k(u_j)], [\eta_k(g), A_k]^{\eta_k(u_j)} \mid 1 \leq j \leq b\rangle. \end{aligned}$$

Thus $([\eta_k(g), U_{i,k}] \mid k \geq 0)$ is parametrized by Lemma 61. Since $\mathrm{supp}(a_j)$ and $\mathrm{supp}(c)$ are finite for $1 \leq j \leq i$ and the family $(U'_{i,k} \mid k \geq 0)$ is parametrized by Proposition 72, the result follows by Theorem 62 (i). □

Proof of Theorem 88. By definition $(U_{i,k} \mid k \geq 0)$ is parametrized and by Lemma 92 the family $(V_k(a,c) \mid k \geq 0)$ is also parametrized. Thus, by Theorem 62 (ii) there exist $j_1, j_2 \in \{0, \dots, d\}$ and rational numbers α_1, α_2 such that $[G_k : U_{i,k}] = \alpha_1 p^{j_1 k}$ and $[G_k : V_k(a,c)] = \alpha_2 p^{j_2 k}$ for all but finitely many $k \geq 0$. Since $[U_{i,k} : V_k(a,c)]$ equals $[G_k : V_k(a,c)]/[G_k : U_{i,k}]$, the result follows. □

Part III

Automorphism groups of finite p-groups by coclass

Chapter 9

Automorphism groups of finite p-groups by coclass

In this chapter the automorphism groups of groups in a coclass family are considered. As introductory example, we discuss the automorphism groups of the dihedral groups of order 2^{k+2} with $k \geq 0$. We write $D_{2^{k+2}}$ and D_∞ to denote the groups

$$D_{2^{k+2}} = C_2 \ltimes \mathbb{Z}_2/2^{k+1}\mathbb{Z}_2 \text{ and } D_\infty = C_2 \ltimes \mathbb{Z}_2,$$

where $C_2 = \langle a \rangle$ acts on $\mathbb{Z}_2$ via $i.a = -i$ for $i \in \mathbb{Z}_2$. Recall that $(D_{2^{k+2}} \mid k \geq 0)$ is a coclass family of coclass 2. Further, for a unit $i \in \mathbb{Z}_2^*$ and for $j \in \mathbb{Z}_2$ let ψ_i and β_j be the group homomorphisms $D_\infty \to D_\infty$ defined by $(a,0)^{\psi_i} = (a,0)$, $(1,1)^{\psi_i} = (1,i)$ and $(a,0)^{\beta_j} = (a,j)$, $(1,1)^{\beta_j} = (1,1)$. Since $D_{2^{k+2}}$ is naturally isomorphic to $D_\infty/2^{k+1}\mathbb{Z}_2$, the maps ψ_i and β_j induce group homomorphisms $\psi_{i,k}$ and $\beta_{j,k}$ from $D_{2^{k+2}}$ to $D_{2^{k+2}}$. It is folklore that ψ_i and β_j are well-defined group automorphisms and that

$$\operatorname{Aut} D_{2^{k+2}} = \langle \psi_{i,k} \mid i \text{ is odd} \rangle \ltimes \langle \beta_{1,k} \rangle.$$

Evidently, the map from the unit group $\mathbb{Z}_2^*$ to $\langle \psi_i \mid i \in \mathbb{Z}_2^* \rangle$ which maps $i \in \mathbb{Z}_2^*$ to ψ_i is a group isomorphism. Note that $\mathbb{Z}_2^*$ equals $1+2\mathbb{Z}_2$ and that $\mathbb{Z}_2$ is a valuation ring of the field $\mathbb{Q}_2$. Further, the group $\langle \beta_j \mid j \in \mathbb{Z}_2 \rangle$ is naturally isomorphic to $\mathbb{Z}_2$. The isomorphisms $\langle \psi_i \mid i \in 1+2\mathbb{Z}_2 \rangle \cong 1+2\mathbb{Z}_2$ and $\langle \beta_j \mid j \in \mathbb{Z}_2 \rangle \cong \mathbb{Z}_2$ give rise to a group isomorphism from the inner semidirect product $A := \langle \psi_i \mid i \in 1+2\mathbb{Z}_2 \rangle \ltimes \langle \beta_j \mid j \in \mathbb{Z}_2 \rangle \leq \operatorname{Aut} D_\infty$ to the outer semidirect product $(1+2\mathbb{Z}_2) \ltimes \mathbb{Z}_2$. For $k \in \mathbb{N}_0$ denote $B_k := \langle \psi_i \mid i \in 1+2^{k+1}\mathbb{Z}_2 \rangle \ltimes \langle \beta_j \mid j \in 2^{k+1}\mathbb{Z}_2 \rangle$. Then it is straightforward to observe that B_k is a well-defined normal subgroup of A and that $\operatorname{Aut} D_{2^{k+2}}$ is isomorphic to A/B_k. For an arbitrary coclass family $(G_k \mid k \geq 0)$ the structure of the automorphism groups $\operatorname{Aut} G_k$ is more complicated but similar to the one of $\operatorname{Aut} D_{2^{k+2}}$.
As usual let $\mathcal{F}(\gamma, \delta, n) = (G_k \mid k \geq 0)$ denote a coclass family and we write P and T to denote the associated point group and translation subgroup, respectively. Let d be the rank of the free $\mathbb{Z}_p$-module T. We further assume that n/d is at least c, where c is a natural number only depending on the residue class of n modulo d, for a concrete definition of c see the paragraph preceding Inequality (9.0.1). The main results of this chapter are the following theorems.

Theorem 93. *There exist a skew field F over $\mathbb{Q}_p$ with discrete valuation ring $\mathcal{O}$, an element $f \in \mathcal{O}$ and a normal subgroup B of finite index in $\operatorname{Aut} E(\gamma)$ such that*

(i) *$[F : \mathbb{Q}_p]$ divides d,*

(ii) *$B \cong (1 + f\mathcal{O}) \ltimes T$.*

For a nonnegative integer k we write B_k to denote the preimage of $(1 + p^k f\mathcal{O}) \ltimes p^k T$ under the isomorphism $B \to (1 + f\mathcal{O}) \ltimes T$ of Theorem 93. Then for $k \geq c$ the factor group B_{k-c}/B_k is abelian and isomorphic to $\mathcal{O}_+/p^c\mathcal{O}_+ \times T/p^cT$, where $\mathcal{O}_+$ denotes the underlying additive group of the ring $\mathcal{O}$.

Theorem 94. *There exist a group $B \leq A \leq \operatorname{Aut} E(\gamma)$, an A-module N and a family $(\mu_k \mid k \geq c)$ of A-module monomorphisms $\mu_k : B_{k-c}/B_k \to N$ such that for $k \geq c$*

(i) *$\operatorname{Aut} G_k$ is isomorphic to a group extension of N by A/B_{k-c},*

(ii) *the image of μ_k is a direct factor of the abelian group N and the diagram*

$$\begin{tikzcd} B_{k-c}/B_k \arrow[r, "\mu_k"] \arrow[d, "\eta_k"] & N \arrow[d, equal, "\mathrm{id}"] \\ B_{k+1-c}/B_{k+1} \arrow[r, "\mu_{k+1}"] & N \end{tikzcd}$$

commutes, where η_k is the group isomorphism sending bB_k to b^pB_{k+1}.

By definition of a coclass family, all groups G_k can be described by a single parametrized presentation. We will show that a similar result holds for the automorphism groups. For this purpose, we introduce group monomorphisms for $k \geq c$

$$\begin{tikzcd} H^2(A/B_0, N) \arrow[r, "\lambda_k"] & H^2(A/B_{k-c}, N) \\ & H^2(A/B_{k-c}, B_{k-c}/B_k) \arrow[u, "\kappa_k"] \end{tikzcd} ,$$

where λ_k is induced by the natural homomorphism $A/B_{k-c} \twoheadrightarrow A/B_0$ and κ_k is induced by the A-module monomorphism $\mu_k : B_{k-c}/B_k \to N$ of Theorem 94. Let $\varrho_k \in H^2(A/B_{k-c}, B_{k-c}/B_k)$ be induced by a transversal map $A/B_{k-c} \to A/B_k$. In particular the group extension defined by ϱ_k is isomorphic to A/B_k.

Theorem 95. *There is $\tau \in H^2(A/B_0, N)$ such that for every $k \geq c$ the group extension defined by $\kappa_k(\varrho_k) + \lambda_k(\tau)$ is isomorphic to $\operatorname{Aut} G_k$.*

Automorphism groups of groups in a coclass family

Let $\mathcal{F} = \mathcal{F}(\gamma, \delta, n) = (G_k \mid k \geq 0)$ be a coclass family and let P and T be such that $\gamma \in Z^2(P, T)$. Further, let d be the rank of the free $\mathbb{Z}_p$-module T, let $T = T_0 \geq T_1 \geq T_2 \geq \cdots$ be the lower P-central series of T and let A_k denote T/T_{n+kd} for k with $n + kd \geq 0$. Recall that we write the $\mathbb{Z}_p$-module T additively and that G_k is the group extension of A_k by P defined by the cocycle $\delta_k := \gamma_{A_k} + p^k\delta_{A_k}$.

As in Subsection 5.3.1 we write Γ and Γ_k to denote the group of compatible pairs $\mathrm{Comp}(P,T)$ and $\mathrm{Comp}(P,A_k)$, respectively. The automorphism group of a 2-cocycle is defined as in Chapter 3. In particular we have

$$\begin{aligned}\operatorname{Aut}\gamma &= \{(\beta,\epsilon,\psi) \mid (\beta,\epsilon)\in\Gamma \text{ and } \psi\in C^1(P,T) \text{ with } \alpha(\psi)=\gamma^{(\beta,\epsilon)}-\gamma\},\\ \operatorname{Aut}\delta_k &= \{(\beta,\epsilon,\psi) \mid (\beta,\epsilon)\in\Gamma_k \text{ and } \psi\in C^1(P,A_k) \text{ with } \alpha_k(\psi)=\delta_k^{(\beta,\epsilon)}-\delta_k\},\end{aligned}$$

where $\alpha : C^1(P,T)\to C^2(P,T)$ and $\alpha_k : C^1(P,A_k)\to C^2(P,A_k)$ denote the coboundary homomorphisms. Recall that the group multiplications in $\operatorname{Aut}\gamma$ and $\operatorname{Aut}\delta_k$ are defined by

$$(\beta_1,\epsilon_1,\psi_1)(\beta_2,\epsilon_2,\psi_2) = (\beta_1\beta_2,\epsilon_1\epsilon_2,\psi_1^{(\beta_2,\epsilon_2)}+\psi_2).$$

Lemma 96. *For $k\geq 1$ the automorphism groups $\operatorname{Aut}E(\gamma)$ and $\operatorname{Aut}G_k$ are isomorphic to $\operatorname{Aut}\gamma$ and $\operatorname{Aut}\delta_k$, respectively.*

Proof. For $k\geq 1$ the groups $E(\gamma)$ and G_k have coclass $\operatorname{cc}P$ by definition of coclass families. Hence the images of the natural embeddings $T\hookrightarrow E(\gamma)$ and $A_k\hookrightarrow G_k$ are characteristic in $E(\gamma)$ and G_k, respectively. Lemma 14 yields the result. $\square$

As in Section 5.3, let $t_0\in T$ be such that $\langle t_0,T_1\rangle = T$, let a be the maximum of $\exp H^2(P,T)$ and $\exp H^3(P,T_n)$, and let b be $\exp H^1(P_{t_0},T_n)$. Further, let c denote the natural number with $p^{c-1}=\max\{a,b\}^2$. Note that a and b do not depend on the choice of the representative $n+d\mathbb{N}_0$. Hence we may assume

$$p^{n/d}\geq p^c>\max\{a,b\}^2 \text{ and } \delta\in C^2(P,T_{n-\log_p(a)\cdot d}) \tag{9.0.1}$$

by Lemma 41. The inequality $p^{n/d}>\max\{a,b\}^2$ implies that $p^{n/d}/\max\{a,b\}$ is at least equal to the exponent of the group $H^2(P,T_n)$: let $m\in\mathbb{N}_0$ be such that $n\leq md<n+d$ and note that $p\cdot Z^2(P,T_n)$ equals $Z^2(P,T_{n+d})\leq Z^2(P,T_{md})$. Thus the group $(p\cdot\exp H^2(P,T_{md}))\cdot Z^2(P,T_n)$ lies in $B^2(P,T_{md})$. Since $B^2(P,T_{md})$ is a subgroup of $B^2(P,T_n)$, we have $p\cdot\exp H^2(P,T_{md})\geq \exp H^2(P,T_n)$. As $H^2(P,T_{md})=H^2(P,p^mT)$ is naturally isomorphic to $H^2(P,T)$, it follows that $p\cdot\max\{a,b\}\geq\exp H^2(P,T_n)$. Hence Assumption (9.0.1) yields

$$p^{n/d}\geq\max\{a,b\}\cdot\exp H^2(P,T_n). \tag{9.0.2}$$

For a map ψ which lies either in $\operatorname{End}_P T$, in $C^1(P,T)$ or in $C^2(P,T)$ we write ψ_{A_k} to denote the map induced by the natural homomorphism and for $k\geq 0$ we define the group homomorphisms

$$\begin{array}{llll}\phi_k: & \operatorname{Aut}\gamma\to\operatorname{Aut}\gamma_{A_k}, & (\beta,\epsilon,\psi)\mapsto(\beta,\epsilon_{A_k},\psi_{A_k}) & \text{and}\\ \phi_{c,k}: & \operatorname{Aut}\delta_k\to\operatorname{Aut}\gamma_{A_{k-c}}, & (\beta,\epsilon,\psi)\mapsto(\beta,\epsilon_{A_{k-c}},\psi_{A_{k-c}}). & \end{array}$$

Evidently, the map ϕ_k is well-defined. By Assumption (9.0.1) the 2-cochain δ is an element of $C^2(P,T_{n-\log_p(a)\cdot d})$ and c is at least $\log_p(a)$. Thus $(\delta_k)_{A_{k-c}}=(\gamma_{A_k}+p^k\delta_{A_k})_{A_{k-c}}$ equals $\gamma_{A_{k-c}}$ and the group homomorphism $\phi_{c,k}$ is well-defined.
In the following sections we study the kernels and images of ϕ_k and $\phi_{k,c}$. Note that the kernel of the projection $\operatorname{Aut}\gamma\to\Gamma$, $(\beta,\epsilon,\psi)\mapsto(\beta,\epsilon)$ is naturally isomorphic to $Z^1(P,T)$ and that $(1,\beta,\psi)\in\operatorname{Aut}\gamma$ implies $\beta\in\operatorname{End}_P T$. Hence it is essential to understand the P-endomorphisms and 1-cocycles in order to describe the kernel of ϕ_k.

9.1 1-Cocycles

Write $\vartheta_{1,k}$ to denote the group homomorphism $Z^1(P,T) \to Z^1(P, A_k)$, $\gamma \mapsto \vartheta_k \circ \gamma$, where ϑ_k is the natural homomorphism $T \twoheadrightarrow A_k$. Further, let $\delta_{1,k} : Z^1(P, A_k) \to H^2(P, T_{n+kd})$ be the connecting homomorphism arising from the Snake Lemma, see Lemma 7 and Sequence (2.2.2).

Lemma 97.

(i) *For $m \in \mathbb{N}_0$ we have $B^1(P, T_m) = Z^1(P, T_{m+1})$, in particular $B^1(P, T_{d-1}) = p \cdot Z^1(P, T)$.*

(ii) *For every $k \geq 0$ the sequence*

$$0 \to \operatorname{Im} \vartheta_{1,k} \hookrightarrow Z^2(P, A_k) \xrightarrow{\delta_{1,k}} H^2(P, T_{n+kd}) \to 0$$

is a split short exact sequence.

Proof. ad (i): For $t \in T$ let δ_t denote the map $P \to T$, $g \mapsto t.g - t$. Then $B^1(P,T)$ equals $\langle \delta_t \mid t \in T \rangle$ by definition. Since P acts uniserially on $T \cong \mathbb{Z}_p^d$ with lower P-central series $T = T_0 \geq T_1 \geq T_2 \geq \cdots$, we have $[P, T_i] = T_{i+1}$ and $T_{i+d} = p \cdot T_i$. This yields $B^1(P, T_i) \subseteq Z^1(P, T_{i+1})$ and $B^1(P, T_i) \not\subseteq Z^1(P, T_{i+2})$ for nonnegative integers i. Let p^f be the exponent of $H^1(P, T_{m+1})$. It follows that $p^f \cdot Z^1(P, T_{m+1}) = B^1(P, T_{m+1}) \cap Z^1(P, T_{fd+m+1}) = B^1(P, T_{fd+m})$. As $\delta_{p \cdot t} = p \cdot \delta_t$ for $t \in T$, we have $B^1(P, T_{fd+m}) = p^f \cdot B^1(P, T_m)$ and thus $p^f Z^1(P, T_{m+1}) = p^f B^1(P, T_m)$. The result follows.

ad (ii): By Assumption (9.0.2) we have $p^{n/d} \geq \exp H^2(P,T)$. This yields $T_n \leq \exp H^2(P,T) \cdot T$. We can now accomplish the proof by applying Lemma 9.

□

9.2 Discrete valuation on $\operatorname{End}_P T$ and the $\operatorname{End}_P T$-module $C^1(P,T)$

We have already seen in Lemma 47 that $\operatorname{End}_P T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a skew field. It will turn out that $\operatorname{End}_P T$ is isomorphic to a valuation ring of $\operatorname{End}_P T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let ν_p be the p-adic valuation and for $i \in \{1, 2\}$ define

$$\begin{aligned} \omega_i : &\quad C^i(P,T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}, \quad \psi \otimes k \mapsto \sup\{m \mid \operatorname{Im} \psi \subseteq T_m\} + \nu_p(k) \cdot d, \\ \omega' : &\quad \operatorname{End}_P(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}, \quad \varphi \otimes k \mapsto \sup\{m \mid \operatorname{Im} \varphi \subseteq T_m\} + \nu_p(k) \cdot d. \end{aligned}$$

The group $\operatorname{End}_P(T) \otimes \mathbb{Q}_p$ acts on $C^i(P,T) \otimes \mathbb{Q}_p$ via $(\psi \otimes k).(\varphi \otimes k') = \psi.\varphi \otimes k \cdot k'$ for $\psi \otimes k \in C^i(P,T) \otimes \mathbb{Q}_p$ and $\varphi \otimes k' \in \operatorname{End}_P(T) \otimes \mathbb{Q}_p$.

Lemma 98. *Let $i \in \{1, 2\}$ and ψ, $\eta \in C^i(P,T) \otimes \mathbb{Q}_p$. Then ω_i is well-defined and*

(i) $\omega_i(\psi + \eta) \geq \min\{\omega_i(\psi), \omega_i(\eta)\}$,

(ii) $\omega_i(\psi) = \infty$ *if and only if* $\psi = 0$, *and*

(iii) $\{\psi \in C^i(P,T) \otimes \mathbb{Q}_p \mid \omega_i(\psi) \geq 0\} = C^i(P,T) \otimes 1$.

Proof. Since T_{m+d} equals $p \cdot T_m$ for $m \in \mathbb{N}_0$, the map ω_i is well-defined. Let $\psi', \eta' \in C^i(P,T)$ and $q_\psi, q_\eta \in \mathbb{Q}_p$ be such that $\psi = \psi' \otimes q_\psi$ and $\eta = \eta' \otimes q_\eta$.

ad (ii): Evidently, $\psi = 0$ yields $\omega_i(\psi) = \infty$. Now assume that $\omega_i(\psi) = \infty$. If q_ψ equals 0, then $\psi = 0$. In the case that q_ψ is nonzero we have $\sup\{m \mid \mathrm{Im}\,\psi' \subseteq T_m\} = \infty$ and thus $\mathrm{Im}\,\psi'$ is a subgroup of $\bigcap_{n\in\mathbb{N}_0} T_n = 0$. The result for (ii) follows.
ad (i): We may assume that ψ and η are nonzero, q_ψ equals q_η and $\omega_i(\psi) \geq \omega_i(\eta)$. Then we have $\psi + \eta = (\psi' + \eta') \otimes q_\psi$. Let m^* be $\sup\{m \in \mathbb{N}_0 \mid \mathrm{Im}\,\eta' \subseteq T_m\}$. By our assumption η is nonzero and thus $m^* \in \mathbb{N}_0$ by (ii). Since $\omega_i(\psi) \geq \omega_i(\eta)$, it follows that $\mathrm{Im}\,\psi' \subseteq T_{m^*}$ and $\mathrm{Im}(\psi' + \eta') \subseteq T_{m^*}$. Consequently, $\omega_i(\psi + \eta)$ is at least $\omega_i(\eta)$.
ad (iii): For $\psi = 0$ we have $\omega_i(\psi) = \infty \geq 0$ and $\psi \in C^i(P,T)\otimes 1$. Let us assume that $\psi = \psi' \otimes q_\psi \in C^i(P,T) \otimes \mathbb{Q}_p$ is nonzero with $\omega_i(\psi) \geq 0$. If $\nu_p(q_\psi)$ is nonnegative, then ψ lies in $C^i(P,T)\otimes 1$. Assume that $\nu_p(q_\psi)$ is negative. The inequality

$$\omega(\psi) = \sup\{m \mid \mathrm{Im}\,\psi' \subseteq T_m\} + \nu_p(q_\psi)\cdot d \geq 0$$

yields that $\psi' \cdot q_\psi : P \to T,\ g \mapsto \psi'(g)\cdot q_\psi$ is a well-defined element of $C^i(P,T)$. Then $\psi' \cdot q_\psi \otimes 1$ equals $\psi' \cdot q_\psi \otimes q_\psi \cdot q_\psi^{-1} = \psi' \cdot q_\psi \cdot q_\psi^{-1} \otimes q_\psi = \psi$ and the result follows.

□

In Lemma 100 we show that a similar result holds for ω'. First, we give a definition of discrete valuations on skew fields.

Definition 99. A *discrete valuation* on a skew field is a map $v : K \to \mathbb{Z} \cup \{\infty\}$ satisfying

(i) $v(xy) = v(x) + v(y)$,

(ii) $v(x) = \infty$ if and only if $x = 0$, and

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$

for every $x, y \in K$. The associated discrete valuation ring is defined as $\{x \in K \mid v(x) \geq 0\}$.

Recall that $\mathrm{End}_P(T) \otimes \mathbb{Q}_p$ is a skew field by Lemma 47.

Lemma 100. *The map ω' is a well-defined discrete valuation on the skew field $\mathrm{End}_P\, T \otimes \mathbb{Q}_p$ with valuation ring $\mathrm{End}_P\, T \otimes 1$. Further, for $i \in \{1,2\}$, $\psi \in C^i(P,T) \otimes \mathbb{Q}_p$ and $\varphi \in \mathrm{End}_P\, T \otimes \mathbb{Q}_p$ we have $\omega_i(\psi.\varphi) = \omega_i(\psi) + \omega'(\varphi)$.*

Proof. Similar to the proof of Lemma 98, we can show an analogous version of Lemma 98 for ω'. Hence it remains to verify the equalities $\omega'(\varphi \cdot \tilde{\varphi}) = \omega'(\varphi) + \omega'(\tilde{\varphi})$ and $\omega_i(\psi.\varphi) = \omega_i(\psi) + \omega'(\varphi)$ for $\varphi, \tilde{\varphi} \in \mathrm{End}_P\, T \otimes \mathbb{Q}_p$ and $\psi \in C^i(P,T) \otimes \mathbb{Q}_p$. Denote $\psi' \otimes q_\psi = \psi$ and $\varphi' \otimes q_\varphi = \varphi$. If either ψ or φ is zero, then $\psi.\varphi$ is trivial and thus $\omega_i(\psi.\varphi) = \infty = \omega_i(\psi) + \omega'(\varphi)$. Assume that ψ and φ are nonzero and let m^* be $\max\{m \mid \mathrm{Im}\,\varphi' \subseteq T_m\}$. Since φ is a P-endomorphism and P acts uniserially on T, the image of φ is T_{m^*}. By induction on $l \in \mathbb{N}$ it follows that

$$T_l.\varphi' = [P, T_{l-1}].\varphi' = [P, T_{l-1}.\varphi'] = [P, T_{l+m^*-1}] = T_{l+m^*}.$$

This yields $\max\{l \mid \mathrm{Im}(\psi'.\varphi') \leq T_l\} = \max\{l \mid \mathrm{Im}\,\psi' \leq T_l\} + m^*$ and thus $\omega_i(\psi.\varphi) = \omega_i(\psi) + \omega'(\varphi)$. The other equality $\omega'(\varphi \cdot \tilde{\varphi}) = \omega'(\varphi) + \omega'(\tilde{\varphi})$ follows similarly. □

Lemma 101. *Let m be a nonnegative integer with $m\mathbb{Z} = \mathrm{Im}\,\omega' \setminus \{\infty\}$. Then m divides d and $[\mathrm{End}_P\, T \otimes \mathbb{Q}_p : \mathbb{Q}_p] = d/m$.*

Proof. Since $d = \omega'(1 \otimes p)$ lies in $\operatorname{Im} \omega'$, the integer m divides d. Evidently, the dimension of the $\mathbb{Q}_p$-vector space $\operatorname{End}_P T \otimes \mathbb{Q}_p$ equals the rank of the free $\mathbb{Z}_p$-module $\operatorname{End}_P T$. Let $\tau \in \operatorname{End}_P T$ be such that $\omega'(\tau \otimes 1) = m$. We show that

$$\bigoplus_{0 \leq i < d/m} \mathbb{Z}_p \cdot \tau^i = \operatorname{End}_P T.$$

For $0 \leq j < d/m$ we have $\omega'(\mathbb{Z}_p \tau^j \otimes 1) = m \cdot j + d\mathbb{Z} \cup \{\infty\}$ and hence $\mathbb{Z}_p \cdot \tau^j$ intersects trivially with $\bigoplus_{0 \leq i < d/m, i \neq j} \mathbb{Z}_p \cdot \tau^i$. It remains to show that $\sum_{0 \leq i < d/m} \mathbb{Z}_p \cdot \tau^i = \operatorname{End}_P T$. Let $\varphi \neq 0$ be an arbitrary element of $\operatorname{End}_P T$. Recall that t_0 denotes an element of T with $\langle t_0, T_1 \rangle = T$ and that $\eta : \operatorname{End}_P T \to T$, $\epsilon \mapsto t_0.\epsilon$ is a $\mathbb{Z}_p$-module monomorphism by Lemma 48. Let $l \in \mathbb{N}_0$ be such that $t_0.\varphi \in T_l \setminus T_{l+1}$. Since φ is a P-endomorphism, we have $\operatorname{Im} \varphi \leq T_l$ and hence $\omega'(\varphi \otimes 1) = l$. This yields $l \in m\mathbb{N}_0$. As $[T_l : T_{l+1}]$ equals p, there exists $1 \leq z < p$ with $t_0.(\varphi - z \cdot \tau^{l/m}) \in T_{l+m}$. Now arguing by induction on l there exists a sequence $z_0, z_1, \ldots$ with $0 \leq z_i < p$ such that $t_0.(\varphi - \sum_{i=0}^{j} z_i \tau^i) \in T_{(j+1)\cdot m}$ for every $j \geq 0$. This yields $\varphi = \sum_{i=0}^{d/m-1} \left(\sum_{l=0}^{\infty} p^l \cdot y_{i,l} \right) \cdot \tau^i$ for some $0 \leq y_{i,l} < p$. The result follows. □

As on page 69 let α denote the coboundary map $C^1(P,T) \to B^2(P,T)$ which maps a 1-cochain map $\psi \in C^1(P,T)$ to the 2-cocycle $\alpha(\psi) : P \times P \to T$, $(g,h) \mapsto \psi(g).h + \psi(h) - \psi(gh)$. Define

$$\alpha \otimes \mathrm{id} : C^1(P,T) \otimes \mathbb{Q}_p \to B^2(P,T) \otimes \mathbb{Q}_p, \ \psi \otimes k \mapsto \alpha(\psi) \otimes k.$$

Lemma 102. *The map $\alpha \otimes \mathrm{id}$ is an $\operatorname{End}_P(T) \otimes \mathbb{Q}_p$-module homomorphism.*

Proof. Let $\psi \otimes k \in C^1(P,T)$ and $\varphi \otimes q \in \operatorname{End}_P(T) \otimes \mathbb{Q}_p$. Further, let g and h be elements of P. Since φ is a P-endomorphism, we have $\psi(g).h.\varphi = \psi(g).\varphi.h$ and thus $\alpha(\psi)(g,h).\varphi = (\psi(g).h + \psi(h) - \psi(gh)).\varphi = \alpha(\psi.\varphi)$. It follows that $(\alpha(\psi) \otimes k).(\varphi \otimes q)$ equals $\alpha(\psi.\varphi) \otimes kq$ and we obtain the result. □

Write o_γ to denote the order of $\gamma + B^2(P,T)$ and let $\psi_\gamma \in C^1(P,T)$ be such that $\alpha(\psi_\gamma) = o_\gamma \cdot \gamma$ and

$$\omega_1(\psi_\gamma \otimes 1) = \max\{\omega_1(\psi \otimes 1) \mid \psi \in C^1(P,T) \text{ with } \alpha(\psi) = o_\gamma \cdot \gamma\}.$$

Corollary 103. *Let $\varphi \in \operatorname{End}_P T$. Then $\psi_\gamma.\varphi$ is of maximal value $\omega_1(\psi_\gamma.\varphi \otimes 1)$ with $\alpha(\psi_\gamma.\varphi) = o_\gamma \gamma.\varphi$.*

Proof. We may assume that ψ_γ and φ are nontrivial. Lemma 102 yields $\alpha(\psi_\gamma.\varphi) = \alpha(\psi_\gamma).\varphi = o_\gamma \gamma.\varphi$. Now assume that there exists $\eta \in \operatorname{End}_P T$ with $\omega_1(\eta \otimes 1) > \omega_1(\psi_\gamma.\varphi \otimes 1)$ and $\alpha(\eta) = o_\gamma \gamma.\varphi$. Denote $\tilde{\eta} := (\eta \otimes 1) \cdot (\varphi \otimes 1)^{-1}$. Then it follows by Lemma 100 and Lemma 102 that $(\alpha \otimes \mathrm{id})(\tilde{\eta}) = o_\gamma \gamma \otimes 1$ and

$$\omega_1(\tilde{\eta}) = \omega_1(\eta \otimes 1) - \omega'(\varphi \otimes 1) > \omega_1(\psi_\gamma.\varphi \otimes 1) - \omega'(\varphi \otimes 1) = \omega_1(\psi_\gamma \otimes 1) \geq 0.$$

Then $\tilde{\eta}$ lies in $C^1(P,T) \otimes 1$ by Lemma 98 (iii). Thus there exists an element $\psi' \in C^1(P,T)$ with $\alpha(\psi') = o_\gamma \gamma$ and $\omega_1(\psi' \otimes 1) > \omega_1(\psi_\gamma \otimes 1)$. This contradicts the construction of ψ_γ and the result follows. □

9.3 Proof of Theorem 93

Corollary 103 enables us to describe $\{\epsilon \mid (\beta,\epsilon,\theta) \in \operatorname{Ker}\phi_l\}$. Let $f \in \operatorname{End}_P T$ denote a P-endomorphism of minimal value $\omega'(f\otimes 1)$ with

$$\omega'(f\otimes 1) \geq n + \log_p(o_\gamma)\cdot d - \omega_1(\psi_\gamma \otimes 1).$$

Recall that $\operatorname{cc} E(\gamma) = \operatorname{cc} P$ by definition of coclass families. Then Lemma 35 yields $\omega_2(\gamma\otimes 1) = 0$ and $\gamma \notin B^2(P,T)$. It follows that $\omega_1(\psi_\gamma\otimes 1) \leq \log_p(o_\gamma)\cdot d$ and hence

$$\omega'(f\otimes 1) \geq n.$$

Corollary 104. *For $k \in \mathbb{N}_0$ the image of the projection $\operatorname{Ker}\phi_k \to \operatorname{End}_P T$, $(\beta,\epsilon,\psi)\mapsto \epsilon$ equals $1 + p^k f \operatorname{End}_P T$.*

Proof. Evidently, for $k\in\mathbb{N}_0$ the map $\operatorname{Ker}\phi_0 \to \operatorname{Ker}\phi_k$, $(1, 1+\varphi, \psi) \mapsto (1, 1+p^k\varphi, p^k\psi)$ is a well-defined bijection. Hence it suffices to prove Corollary 104 for $k=0$. Let $(1,1+\varphi,\psi)$ be an element of $\operatorname{Ker}\phi_0$. Since $\alpha(o_\gamma\psi) = o_\gamma\gamma.\varphi$, it follows by Corollary 103 that

$$\omega_1(\psi_\gamma.\varphi\otimes 1) \geq \omega_1(\psi\otimes o_\gamma) \geq n + \log_p(o_\gamma)\cdot d.$$

Then Lemma 100 yields $\omega'(\varphi\otimes 1) \geq n + \log_p(o_\gamma)d - \omega_1(\psi_\gamma\otimes 1)$, in particular $\omega'(\varphi\otimes 1)\geq \omega'(f\otimes 1)$. Hence we have $\omega'((\varphi\otimes 1)(f\otimes 1)^{-1}) \geq 0$, in other words $\varphi \in f\cdot \operatorname{End}_P T$. Since the image of $o_\gamma^{-1}\psi_\gamma.f$ is a subgroup of T_n, the element $(1, 1+f\cdot\varphi', o_\gamma^{-1}\cdot\psi_\gamma.(f\cdot\varphi'))$ lies in $\operatorname{Ker}\phi_0$ for every $\varphi'\in\operatorname{End}_P T$. The result follows. □

We write ι_k and ϑ_k to denote the group homomorphisms

$$\begin{array}{rccc} \iota_k: & p^kT_{n-1} & \to & \operatorname{Ker}\phi_k, \\ & t & \mapsto & (1,1,\delta_t) \end{array} \qquad \begin{array}{rccc} \vartheta_k: & 1+p^kf\operatorname{End}_P T & \to & \operatorname{Ker}\phi_k, \\ & 1+\varphi & \mapsto & (1, 1+\varphi, o_\gamma^{-1}\psi_\gamma.\varphi) \end{array}$$

where δ_t denotes the coboundary $P\to T$, $g\mapsto t.g - t$. Further, let $\vartheta_k \ltimes \iota_k$ be such that the following diagram commutes:

$$\begin{tikzcd} p^kT_{n-1} \arrow[r] \arrow[dr, "\iota_k"'] & (1+p^kf\operatorname{End}_P T)\ltimes p^kT_{n-1} \arrow[d, "\vartheta_k\ltimes\iota_k"] & 1+p^kf\operatorname{End}_P T \arrow[l] \arrow[dl, "\vartheta_k"] \\ & \operatorname{Ker}\phi_k & \end{tikzcd}$$

Lemma 105. *The maps ι_k and ϑ_k are well-defined group monomorphisms and the semidirect product $(1+p^kf\operatorname{End}_P T)\ltimes p^kT_{n-1}$ is isomorphic to $\operatorname{Ker}\phi_k$ via $\vartheta_k\ltimes\iota_k$.*

Proof. Fist, note that $\operatorname{Im}\iota_k = \{(1,1,\psi) \mid \psi \in B^1(P, T_{n+kd-1})\}$. By Lemma 97 the group $B^1(P,T_{n+kd-1})$ is $Z^1(P,T_{n+kd})$ and thus $\operatorname{Im}\iota_k \leq \operatorname{Ker}\phi_k$. It is straightforward to check that ι_k and ϑ_k are group homomorphisms. Since the coboundary map $T\to Z^1(P,T)$, $t\mapsto\delta_t$ is a $\operatorname{End}_P T$-module homomorphism, we have $\iota_k(t.\varphi) = \iota_k(t)^{\vartheta_k(\varphi)}$ for $t\in p^kT_{n-1}$ and $\varphi \in 1+p^kf\operatorname{End}_P T$. Thus $\vartheta_k\ltimes\iota_k$ is a well-defined group homomorphism. It remains to show that $\vartheta_k\ltimes\iota_k$ is bijective. Evidently, the map ϑ_k is injective. Let $t\in p^kT_{n-1}$ be such that $\iota_k(t) = 0$, in particular $\delta_t = 0$. This is equivalent to $t \in C_T(P)$. Assume that t is nontrivial and let $e\in\mathbb{N}_0$ be such that $t\in T_e\setminus T_{e+1}$. It follows by $[T_e : T_{e+1}] = p$ that $\langle t, T_{e+1}\rangle = T_e$. Since $[T_{e+1},P] = T_{e+2}$ and $t\in C_T(P)$, we have $[T_e, P] = T_{e+2}$. This contradicts $T_{e+1} = [T_e, P]$ and thus $C_T(P) = 0$. Hence ι_k is injective. Since $\operatorname{Im} \vartheta_k$ intersects trivially with $\operatorname{Im} \iota_k$, the homomorphism $\vartheta_k \ltimes \iota_k$ is injective. Now, we show that $\vartheta_k \ltimes \iota_k$ is also surjective. Let $(1, 1 + \varphi, \psi)$ be an element of the kernel of ϕ_k. Then φ lies in $p^k f \operatorname{End}_P T$ by Lemma 104. Further, $\psi' := \psi - o_\gamma^{-1} \psi_\gamma.\varphi$ is an element of $Z^1(P, p^k T_n)$ and by Lemma 97 there exists $t \in p^k T_{n-1}$ such that $\delta_t = \psi'$. This yields $\vartheta_k(\varphi) \cdot \iota_k(t) = (1, 1 + \varphi, \psi)$ and thus $\vartheta_k \ltimes \iota_k$ is surjective. □

Proof of Theorem 93. As $\operatorname{Im} \phi_0$ is finite, the kernel of ϕ_0 is a normal subgroup of $\operatorname{Aut} \gamma$ of finite index. By Lemma 105 the group $\operatorname{Ker} \phi_0$ is isomorphic to the semidirect group $(1+f \operatorname{End}_P T) \ltimes T_{n-1}$, where f is an element of $\operatorname{End}_P T$. Further, $\operatorname{End}_P T \otimes 1$ is a discrete valuation ring of the skew field $F := \operatorname{End}_P T \otimes \mathbb{Q}_p$ and the dimension of the $\mathbb{Q}_p$-vector space divides d by Lemmata 100 and 101. Since $\operatorname{Aut} E(\gamma)$ is isomorphic to $\operatorname{Aut} \gamma$ by Lemma 96, the result follows. □

9.4 Proofs of Theorems 94 and 95

In this section we apply the results from the previous sections in order to describe $\operatorname{Aut} \delta_k$ as group extension of $\operatorname{Ker} \phi_{c,k}$ by $\operatorname{Im} \phi_{c,k}$ for $k \geq c$. This enables us to prove Theorems 94 and 95.
In the following two lemmata we consider the image and kernel of $\phi_{c,k}$. Let E be as in the paragraph preceding Lemma 51. In particular E is an additive group of maps $T \to T$ and for $E_k := (p^k E)_{A_k}$ we have $\operatorname{End}_P A_k = (\operatorname{End}_P T) \oplus E_k$. Denote

$$B_k := \operatorname{Ker} \phi_k.$$

Similar to $\operatorname{Aut} \delta_k$ we define a group structure on the set $\Gamma_k \times C^1(P, A_k)$ by setting $(\beta, \eta, \psi) \cdot (\beta', \eta', \psi') := (\beta\beta', \eta\eta', \psi.\eta' + \psi')$.

Lemma 106. *For nonnegative integers $k \geq l$ let $f_{k,l} : \operatorname{Im} \phi_{c,k} \to \operatorname{Im} \phi_{c,l}$ be defined by setting $f_{k,l}(\beta, \epsilon, \psi) = (\beta, \epsilon_{A_{l-c}}, \psi_{A_{l-c}})$. Then:*

- (i) *The family $(f_{k,l} \mid k \geq l \geq 0)$ is an inverse family of homomorphisms. Let A be the inverse limit $\varprojlim \operatorname{Im} \phi_{c,k}$. Then A can be regarded as subgroup of $\operatorname{Aut} \gamma$ with $B_0 \leq A$.*
- (ii) *The image of B_0 under ϕ_k is a subgroup of $\operatorname{Aut} \delta_k$. Further, there exists a map $\chi = (\chi_1, \chi_2) : A/B_0 \to E \times C^1(P, T)$ such that for $g = (\beta, \epsilon, \psi) \cdot B_0 \in A/B_0$ the product of $\phi_k(\beta, \epsilon, \psi) = (\beta, \epsilon_{A_k}, \psi_{A_k})$ and $(1, 1 + p^k \chi_1(g)_{A_k}, p^k \chi_2(g)_{A_k})$ lies in $\operatorname{Aut} \delta_k$ for every $k \geq 0$.*

Proof. Let k denote a nonnegative integer and let $(\beta, \epsilon_k, \psi_k)$ be an element of $\operatorname{Aut} \delta_k$. We show that there exist $(\beta, \epsilon, \psi) \in \operatorname{Aut} \gamma$, a map $\varphi \in E$ and $\psi' \in C^1(P, T)$ such that $(\beta, \epsilon_{A_k}, \psi'_{A_k}) \cdot (1, 1 + p^k \varphi_{A_k}, p^k \psi_{A_k})$ equals $(\beta, \epsilon_k, \psi_k)$, and for every $l \in \mathbb{N}_0$ the product $(\beta, \epsilon_l, \psi_l) := (\beta, \epsilon_{A_l}, \psi_{A_l}) \cdot (1, 1 + p^l \varphi_{A_l}, p^l \psi'_{A_l})$ lies in $\operatorname{Aut} \delta_l$ and its image under $\phi_{c,l}$ is $(\beta, \epsilon_{A_{l-c}}, \psi_{A_{l-c}})$.
By definition of $\operatorname{Aut} \delta_k$ the pair (β, ϵ_k) is a compatible pair, in other words $(\beta, \epsilon_k) \in \Gamma_k = \operatorname{Comp}(P, A_k)$. By Lemma 51 there exist $(\beta, \epsilon) \in \Gamma = \operatorname{Comp}(P, T)$ and a map $\varphi \in E$ such that for $\varphi_k := p^k \varphi_{A_k} \in E_k$ the product $(\beta, \epsilon_{A_k})(1 + \varphi_k)$ equals (β, ϵ_k). For $l \in \mathbb{N}_0$ put $\varphi_l := p^l \varphi_{A_l} \in E_l$ and $\epsilon_l := \epsilon_{A_l}(1 + \varphi_l)$. Let $\lambda_l : \Gamma_0 / \operatorname{Im}(\pi_0 \circ \varrho) \to \Gamma_l / \operatorname{Im}(\pi_l \circ \varrho)$ be defined as in the paragraph preceding Lemma 52 and note that the image of $(\beta, \epsilon_0) \cdot \operatorname{Im}(\pi_0 \circ \varrho)$ under λ_l is $(\beta, \epsilon_l) \cdot \operatorname{Im}(\pi_l \circ \varrho)$. By definition of $\operatorname{Aut} \delta_k$ the compatible pair $(\beta, \epsilon_k) \in \Gamma_k$ stabilizes $\delta_k + B^2(P, A_k)$ and hence by Lemma 52 the pair $(\beta, \epsilon_l) \in \Gamma_l$ lies in the stabilizer of $\delta_l + B^2(P, A_l)$. Further, note that by Lemma

49 and by definition of E_l the exponent of E_l coincides with the exponent of $H^1(P_{t_0}, T_n)$, which is b. It follows that

$$(\varphi_l)_{A_{l-\log_p b}} = 0 \quad \text{and} \quad (\epsilon_l)_{A_{l-\log_p b}} = \epsilon_{A_{l-\log_p b}} \tag{9.4.3}$$

for every $l \geq 0$. In particular (β, ϵ) stabilizes $\gamma + B^2(P,T)$, and $(\beta, (\epsilon_l)_{A_{l-c}}) = (\beta, \epsilon_{A_{l-c}})$.
Recall that $\delta_l = \gamma_{A_l} + p^l \delta_{A_l}$ and that $\alpha_l : C^1(P, A_l) \to C^2(P, A_l)$ denotes the coboundary map. Now, we construct $\psi_l \in C^1(P,T)$ with $(\beta, \epsilon_l, \psi_l) \in \operatorname{Aut} \delta_l$; this is equivalent to $\alpha_l(\psi_l) = [\delta_l, (\beta, \epsilon_l)]$. An easy calculation yields $[\delta_l, (\beta, \epsilon_l)] = \delta_{1,l} + \delta_{2,l} + \delta_{3,l}$ with $\delta_{1,l} := [\gamma, (\beta, \epsilon)]_{A_l}$, $\delta_{2,l} := [p^l \delta_{A_l}, (\beta, \epsilon)] + ([\gamma_{A_l}, (\beta, \epsilon)] + \gamma_{A_l}).\varphi_l$ and $\delta_{3,l} := [p^l \delta_{A_l}, (\beta, \epsilon)].\varphi_l + p^l \delta_{A_l}.\varphi_l$. We show that $\delta_{3,l}$ is trivial. First, recall that $\delta \in C^2(P, T_{n-\log_p(a)d})$ by Assumption (9.0.1), and the image of φ_l is a subgroup of $T_{n+(l-\log_p(b))d}$ by Equation (9.4.3). Then Lemma 100 yields $p^l \delta_{A_l}.\varphi_l \in Z^2(P, T_m/T_{n+ld})$, where m denotes the minimum of $n + ld$ and $n + 2ld + (n - (\log_p a + \log_p b)d)$. By assumption (9.0.1) the natural number n is at least $(\log_p a + \log_p b) \cdot d$ and hence m equals $n + ld$ and $p^l \delta_{A_l}.\varphi_l$ is trivial. This yields $[\delta_l, (\beta, \epsilon_l)] = \delta_{1,l} + \delta_{2,l}$. Since (β, ϵ) and (β, ϵ_0) stabilize $\gamma + B^2(P,T)$ and $\delta_0 + B^2(P, A_k)$, respectively, and $\delta_{1,0} = [\gamma, (\beta, \epsilon)]_{A_0}$, there exist ψ, $\psi' \in C^1(P,T)$ with $\alpha(\psi) = [\gamma, (\beta, \epsilon)]$ and $\alpha_0(\psi'_{A_0}) = [\delta_0, (\beta, \epsilon_0)] - \delta_{1,0} = \delta_{2,0}$. Evidently, $\alpha_k(p^k \psi')_{A_k}$ equals $\delta_{2,k}$ and thus the difference of ψ_k and $\psi_{A_k} + p^k \psi'_{A_k}$ lies in $Z^1(P, A_k)$. As $p^{n/d} > \exp H^2(P, T_n) = H^2(P, T_{n+kd})$, Lemma 97 yields the existence of $\eta \in Z^1(P,T)$, $\eta' \in C^1(P,T)$ with $\eta'_{A_0} \in Z^1(P, A_0)$ and $\psi_k = (\psi + \eta)_{A_k} + p^k(\psi'_{A_k} + \eta'_{A_k})$. Without loss of generality we assume that $\eta = \eta' = 0$. Put $\psi_l := \psi_{A_l} + p^l \psi'_{A_l}$. Then $(\beta, \epsilon_l, \psi_l)$ lies in $\operatorname{Aut} \delta_l$.
It remains to show $(\psi_l)_{A_{l-c}} = \psi_{A_{l-c}}$. Let e denote the p-logarithm of $\max\{a, b\}$ and note that $(\beta, \epsilon_0)_{A_{0-e}} = (\beta, \epsilon)_{A_{0-e}}$ by Equation (9.4.3), and $(\delta_0)_{A_{-e}} = \gamma_{A_{-e}}$. This yields $\alpha(\psi')_{A_{-e}} = (\delta_{2,0})_{A_{-e}} = 0$ and thus $\psi' \in Z^1(P, A_{-e})$. As the exponent of $A_{-e} = T/T_{n-ed}$ is at least $p^{c-e} \geq \exp H^2(P, T_n)$ by Equation (9.0.2), the group $Z^1(P, A_{-e})$ is a direct sum of $Z^1(P,T)_{A_{-e}}$ and a subgroup of exponent $\exp H^2(P, T_n)$ by Lemma 97. Thus $(\psi')_{A_{-e-(c-e)}}$ lies in $Z^1(P,T)_{A_{-c}}$ and we may assume that $0 = (p^l \psi')_{A_{l-c}} = (p^k \psi')_{A_{k-c}}$. Summing up, we obtain $(\beta, \epsilon_l, \psi_l) \in \operatorname{Aut} \delta_l$ and $\phi_{c,l}(\beta, \epsilon_l, \psi_l) = (\beta, \epsilon_{A_{l-c}}, \psi_{A_{l-c}})$.
This yields that $(f_{k,l} \mid k \geq l \geq 0)$ is an inverse family whose inverse limit A can be considered as subgroup of $\operatorname{Aut} \gamma$. Further, it follows that there exists a map $\chi' = (\chi'_1, \chi'_2) : A \to E \times C^1(P,T)$ such that for $g = (\beta, \epsilon, \psi)$ the product $(\beta, \epsilon_{A_k}, \psi_{A_k}) \cdot (1, 1 + p^k \chi'_1(g)_{A_k}, p^k \chi'_2(g)_{A_k})$ is an element of $\operatorname{Aut} \delta_k$. In order to prove Lemma 106 it remains to show $B_0 \leq A$ and $\phi_k(B_0) \leq \operatorname{Aut} \delta_k$. Because of $a \geq \exp H^2(P, A_k)$, the group $1 + a \operatorname{End}_P T$ stabilizes $\gamma_{A_k} + B^2(P, A_k) \in H^2(P, A_k)$ and fixes $p^k \delta_{A_k} \in Z^2(P, T_{n+(k-\log_p a)d}/T_{n+kd})$. Since $p^{n/d} \geq a$, the groups $B_0 = \operatorname{Ker} \phi_0$ and $\phi_k(B_0)$ are subgroups of A and $\operatorname{Ker} \delta_k$, respectively. □

It is straightforward to conclude from Assumption (9.0.1) and Lemma 100 that $\operatorname{Ker} \phi_{c,k}$ is abelian for $k \geq c$.

Lemma 107. *Assume that $k \geq c$. Then Diagram 2 on page 76 commutes and has exact columns and rows, and ϕ_k embeds B_{k-c}/B_k as a direct factor in the abelian group* $\operatorname{Ker} \phi_{c,k}$.

Proof. First, note that by Lemma 106 (ii) the image $\phi_k(B_0)$ is a subgroup of $\operatorname{Aut} \delta_k$ and thus $\phi_k(B_{k-c}) \leq \operatorname{Ker} \phi_{c,k}$ for every $k \geq c$. Evidently, Diagram 2 commutes and has exact rows and columns. It remains to show that $\phi_k(B_{k-c})$ is a direct factor of $\operatorname{Ker} \phi_{c,k}$.
Assume that $\phi_k(B_{k-c})$ is not a direct factor of $\operatorname{Ker} \phi_{c,k}$. Then there exist $g \in \operatorname{Ker} \phi_{c,k}$ and a p-power m such that g^m lies in $\phi_k(B_{k-c})$ and there is no element $h \in \phi_k(B_{k-c})$ with $h^m = g^m$, see

$$
\begin{tikzcd}
& & & 0 & \\
0 \arrow[r] & B_{k-c} \arrow[r, hook] \arrow[dd, equal] & A \arrow[r, "\phi_{k-c}"] & \operatorname{Im}\phi_{c,k} \arrow[u] \arrow[r] & 0 \\
& & & \operatorname{Aut}\delta_k \arrow[u, "\phi_{c,k}"'] & \\
B_k \arrow[r, hook] & B_{k-c} \arrow[rr, "\phi_k"] & & \operatorname{Ker}\phi_{c,k} \arrow[u, hook] & \\
& & & 0 \arrow[u] &
\end{tikzcd}
$$

Diagram 2

[27, Exercise 4.5]. Let $(1, 1+\varphi, \psi) \in B_{k-c}$ with $(1, 1+\varphi_{A_k}, \psi_{A_k}) = g^m$. Since $\operatorname{Ker}\phi_{c,k}$ is abelian and g lies in $\operatorname{Ker}\phi_{c,k}$, we have $(1, 1+(m^{-1}\varphi), m^{-1}\psi) \in B_{k-c} = \operatorname{Ker}\phi_{k-c}$. This contradicts the assumption that g^m has no m-th root in $\phi_k(B_{k-c})$. The result follows. $\square$

By Lemma 107 the group A/B_{k-c} is naturally isomorphic to $\operatorname{Aut}\delta_k/\operatorname{Ker}\phi_{c,k}$. This induces an action of A on $\operatorname{Ker}\phi_{c,k}$. We write N to denote $\operatorname{Ker}\phi_{c,c}$ and for $k \geq c$ we define

$$
\begin{array}{lcccccc}
\theta_k: & N & \to & \operatorname{Ker}\phi_{c,k}, & (1, 1+\varphi, \psi) & \mapsto & (1, 1+p^{k-c}\cdot\varphi_{A_k}, p^{k-c}\cdot\psi_{A_k}), \\
\eta_k: & B_{k-c}/B_k & \to & B_{k+1-c}/B_{k+1}, & b\cdot B_k & \mapsto & b^p\cdot B_{k+1}, \\
\mu_k: & B_{k-c}/B_k & \to & N & b\cdot B_k & \mapsto & \theta_k^{-1}\circ\phi_k(b).
\end{array}
$$

Note that for $b = (1, 1+\varphi, \psi) \in B_{k-c}$ we have $\eta_k(b\cdot B_k) = (1, 1+p\cdot\varphi, p\cdot\psi)\cdot B_{k+1}$. It is straightforward to observe

Lemma 108. *The maps θ_k, η_k and μ_k are well-defined A-module homomorphisms. Further θ_k and η_k are bijective, and the following diagram commutes:*

$$
\begin{tikzcd}
B_{k-c}/B_k \arrow[r, "\mu_k"] \arrow[d, "\eta_k"] & N \arrow[d, equal, "\mathrm{id}"] \\
B_{k+1-c}/B_{k+1} \arrow[r, "\mu_{k+1}"] & N
\end{tikzcd}
$$

Now we are able to prove Theorem 94.

Proof of Theorem 94. Theorem 94 follows directly from Lemma 96, Lemma 107 and Lemma 108. $\square$

By Lemma 107 and Lemma 108 the automorphism group $\operatorname{Aut}\delta_k$ is isomorphic to a group extension of the abelian group N by A/B_{k-c}. In what follows, we describe a corresponding 2-cocycle of $\operatorname{Aut}\delta_k$. For $k \geq c$ let $\lambda_k : H^2(A/B_0, N) \to H^2(A/B_{k-c}, N)$ and $\kappa_k : H^2(A/B_{k-c}, B_{k-c}/B_k) \to$

$H^2(A/B_{k-c}, N)$ be as in the paragraph preceding of Theorem 95, that is, λ_k and κ_k are induced by the natural homomorphism $A/B_{k-c} \to A/B_0$ and by the A-module monomorphism $\mu_k : B_{k-c}/B_k \to N$, respectively.

$$\begin{tikzcd} H^2(A/B_0, N) \arrow[r, "\lambda_k"] & H^2(A/B_{k-c}, N) \\ & H^2(A/B_{k-c}, B_{k-c}/B_k) \arrow[u, "\kappa_k"] \end{tikzcd}$$

Let ϱ_k denote an element of $H^2(A/B_{k-c}, B_{k-c}/B_k)$ arising naturally from a transversal map $A/B_{k-c} \to A/B_k$.

Theorem 109. *There exists $\tau \in H^2(A/B_0, N)$ such that for every $k \geq c$ the group extension defined by $\kappa_k(\varrho_k) + \lambda_k(\tau)$ is isomorphic to* $\operatorname{Aut}\delta_k$.

Proof. Let $\nu_{k,1} : A/B_{k-c} \to A$ be a transversal map. Further, let $\chi = (\chi_1, \chi_2) : A/B_0 \to E \times C^1(P, T)$ be as in Lemma 106 and write $\chi_{(k)}$ to denote the map

$$\chi_{(k)} : A/B_{k-c} \to \Gamma_k \times C^1(P, A_k),\ g \mapsto (1, 1 + p^k \cdot \chi_1(gB_0)_{A_k}, p^k \cdot \chi_2(gB_0)_{A_k}).$$

Recall that the group multiplication of $\Gamma_k \times C^1(P, A_k)$ is compatible with the one of $\operatorname{Aut}\delta_k$, that is, $(\beta, \epsilon, \psi) \cdot (\beta', \epsilon', \psi') = (\beta\beta', \epsilon\epsilon', \psi^{(\beta', \epsilon')} + \psi')$. By Lemmata 106 and 107 the map $\nu_{k,2} : A/B_{k-c} \to \operatorname{Aut}\delta_k$, $g \mapsto (\phi_k \circ \nu_{k,1})(g) \cdot \chi_{(k)}(g)$ is well-defined and $\operatorname{Im}\nu_{k,2}$ is a transversal of the factor group $\operatorname{Aut}\delta_k/\operatorname{Ker}\phi_{c,k} \cong A/B_{k-c}$ in $\operatorname{Aut}\delta_k$. Let $\vartheta_{k,1}$ and $\vartheta_{k,2} \in Z^2(A/B_{k-c}, N)$ be the 2-cocycles induced by $\nu_{k,1}$ and $\nu_{k,2}$, that is,

$$\begin{aligned} \vartheta_{k,1} &: (A/B_{k-c})^2 \to N, \quad (g,h) \mapsto (\theta_k^{-1} \circ \phi_k)(\nu_{k,1}(gh)^{-1} \cdot \nu_{k,1}(g) \cdot \nu_{k,1}(h)), \\ \vartheta_{k,2} &: (A/B_{k-c})^2 \to N, \quad (g,h) \mapsto \theta_k^{-1}(\nu_{k,2}(gh)^{-1} \cdot \nu_{k,2}(g) \cdot \nu_{k,2}(h)). \end{aligned}$$

Evidently, the group extension defined by $\vartheta_{k,2}$ is isomorphic to $\operatorname{Aut}\delta_k$. Let τ_k denote the 2-cocycle $\vartheta_{k,2} - \vartheta_{k,1}$. It can easily be verified that the normal closure of the set $\operatorname{Im}\chi_{(k)}$ in $\Gamma_k \times C^1(P, A_k)$ is abelian and centralizes $\operatorname{Ker}\phi_{c,k} \leq \Gamma_k \times C^1(P, A_k)$. This yields $\tau_k(g,h) = \theta_k^{-1}(\chi_{(k)}(gh)^{-1} \cdot \chi_{(k)}(g)^h \cdot \chi_{(k)}(h))$ and $\lambda_k(\tau_c + B^2(A/B_0, N)) = \tau_k + B^2(A/B_{k-c}, N)$. Since τ_k does not depend on the choice of the transversal map $\nu_{k,1}$, we may assume that $\varrho_k = \vartheta_{k,1} + B^2(A/B_{k-c}, N)$. Then we have $\vartheta_{k,2} + B^2(A/B_{k-c}, N) = \varrho_k + \lambda_k(\tau_c + B^2(A/B_0, N))$ and the result follows. □

Proof of Theorem 95. Since $\operatorname{Aut}\delta_k$ is isomorphic to $\operatorname{Aut}G_k$ by Lemma 96, Theorem 109 yields Theorem 95. □

Appendix A

Character degrees of 2-groups of coclass 2

In this chapter we consider the character degree polynomials of the 51 coclass families of 2-groups of coclass 2. For this purpose, we use the parametrized presentations of the coclass families given in [10]. In GAP [11] we have determined the irreducible characters of the first groups of each coclass family. These computer calculations suggest that the character degree polynomials are as listed in Table A.1. The rows of the table correspond to the coclass families which are listed in the column "family". Moreover, the associated pro-2-group of a coclass family can be read off the column "pro-2-group". The rows of coclass families which have the same character degree polynomials as the preceding coclass family are omitted in the table and replaced by vertical dots.
We explain the other notations of the table by the row with the coclass family $\mathcal{K}^{13} = (\mathcal{K}^{13}_k \mid k \geq 0)$. Denote $f_0 := 8$, $f_1 := 2$, $f_2 := -1 + 2^3x^2$ and $f_3 := 2x^2$. Then for $k \geq 0$ and $1 \leq l \leq 3$ the number of irreducible characters of $\mathcal{K}^{13}_k$ of degree 2^l, denoted by $\mathrm{N}_l(\mathcal{K}^{13}_k)$, is $f_l(2^k)$, and the number of conjugacy classes of $\mathcal{K}^{13}_k$ is $9 + 5 \cdot 2(2^k)^2$.

Example 110. We consider the irreducible characters of degree 2^2 of the groups in the coclass families $\mathcal{K}^{36} = (\mathcal{K}^{36}_k \mid k \geq 0)$ and $\mathcal{K}^{39} = (\mathcal{K}^{39}_k \mid k \geq 0)$, which have the same associated pro-2-group S_3. The groups $\mathcal{K}^{36}_k$ and $\mathcal{K}^{39}_k$ are defined by the following group presentations

$$\begin{aligned}
\mathcal{K}^{36}_k = \langle\, g_1, \ldots, g_4, t_1 \mid \;& g_1^2 = g_4,\ g_2^{g_1} = g_2g_3, g_2^2 = 1,\ g_3^{g_1} = g_3t_1, g_3^{g_2} = g_3t_1, \\
& g_3^2 = t_1^{-1},\ g_4^{g_1} = g_4, g_4^{g_2} = g_4, g_4^{g_3} = g_4, g_4^2 = 1, \\
& t_1^{g_1} = t_1^{-1}, t_1^{g_2} = t_1^{-1}, t_1^{2^{k+4}} = 1 \,\rangle \text{ and} \\
\mathcal{K}^{39}_k = \langle\, g_1, \ldots, g_4, t_1 \mid \;& g_1^2 = g_4,\ g_2^{g_1} = g_2g_3,\ g_2^2 = 1,\ g_3^{g_1} = g_3t_1, g_3^{g_2} = g_3t_1^{1+2^{k+3}}, \\
& g_3^2 = t_1^{-1+2^{k+3}},\ g_4^{g_1} = g_4, g_4^{g_2} = g_4t_1^{2^{k+3}}, g_4^{g_3} = g_4, g_4^2 = 1, \\
& t_1^{g_1} = t_1^{-1}, t_1^{g_2} = t_1^{-1}, t_1^{2^{k+4}} = 1 \,\rangle
\end{aligned}$$

for $k \geq 0$. Let $k \in \mathbb{N}_0$ and denote $G := \mathcal{K}^{36}_k$ and $H := \mathcal{K}^{39}_k$. Then $\langle g_1g_2, g_3, g_4, t_1\rangle$ is an abelian normal subgroup of G with index 2 in G. Thus we obtain $\mathrm{N}_2(\mathcal{K}^{36}_k) = 0$ by Lemma 85.
Now, we determine $\mathrm{N}_2(\mathcal{K}^{39}_k) = \mathrm{N}_2(H)$. The normal abelian subgroup $V := \langle g_3, g_4, t_1\rangle$ has index 2^2 in H. Thus every irreducible character of degree 2^2 is linearly induced from V by Lemma 85. Note that $V\backslash H/V$ is $H/V = \{1V, g_1V, g_2V, g_1g_2V\}$. Further, we have $[1, V] = 1$, $[g_1, V] = \langle t_1\rangle$,

pro-p-group	family	N_0	N_1	N_2	N_3	N_4	$\lvert G_k\rvert$	# conjugacy classes
S_1	$\mathcal{K}^1$	8	2	$-1+2^3x^2$	0	0	$2^7 2^{2k}$	$9+2^3x^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^3$	8	2	$-1+2^3x^2$	0	0	$2^7 2^{2k}$	$9+2^3x^2$
	$\mathcal{K}^4$	8	2	$-1+2^2x^2$	x^2	0	$2^7 2^{2k}$	$9+5x^2$
	$\mathcal{K}^5$	8	2	$-1+2^2x^2$	x^2	0	$2^7 2^{2k}$	$9+5x^2$
	$\mathcal{K}^6$	8	2	$-1+2^3x^2$	0	0	$2^7 2^{2k}$	$9+2^3x^2$
	$\mathcal{K}^7$	8	2	$-1+2^3x^2$	0	0	$2^7 2^{2k}$	$9+2^3x^2$
	$\mathcal{K}^8$	8	2	$-1+2^2x^2$	x^2	0	$2^7 2^{2k}$	$9+5x^2$
	$\mathcal{K}^9$	8	2	$-1+2^2x^2$	x^2	0	$2^7 2^{2k}$	$9+5x^2$
	$\mathcal{K}^{10}$	8	2	$-1+2^4x^2$	0	0	$2^8 2^{2k}$	$9+2^4x^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{12}$	8	2	$-1+2^4x^2$	0	0	$2^8 2^{2k}$	$9+2^4x^2$
	$\mathcal{K}^{13}$	8	2	$-1+2^3x^2$	$2x^2$	0	$2^8 2^{2k}$	$9+5\cdot 2x^2$
	$\mathcal{K}^{14}$	8	2	$-1+2^3x^2$	$2x^2$	0	$2^8 2^{2k}$	$9+5\cdot 2x^2$
	$\mathcal{K}^{15}$	8	2	$-1+2^4x^2$	0	0	$2^8 2^{2k}$	$9+2^4x^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{17}$	8	2	$-1+2^4x^2$	0	0	$2^8 2^{2k}$	$9+2^4x^2$
	$\mathcal{K}^{18}$	8	2	$-1+2^3x^2$	$2x^2$	0	$2^8 2^{2k}$	$9+5\cdot 2x^2$
	$\mathcal{K}^{19}$	8	2	$-1+2^3x^2$	$2x^2$	0	$2^8 2^{2k}$	$9+5\cdot 2x^2$
S_2	$\mathcal{K}^{20}$	8	2	$-5+2^2\cdot 3x$	$1-3x+2x^2$	0	$2^7 2^{2k}$	$6+3^2x+2x^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{23}$	8	2	$-5+2^2\cdot 3x$	$1-3x+2x^2$	0	$2^7 2^{2k}$	$6+3^2x+2x^2$
	$\mathcal{K}^{24}$	8	2	$-5+2^4x$	$1-2^2x+2^2x^2$	0	$2^8 2^{2k}$	$6+2^2\cdot 3x+2^2x^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{31}$	8	2	$-5+2^4x$	$1-2^2x+2^2x^2$	0	$2^8 2^{2k}$	$6+2^2\cdot 3x+2^2x^2$
	$\mathcal{K}^{32}$	8	2	$-5+3\cdot 2^2x$	$1-x+2x^2$	$-2^{-1}x+2^{-1}x^2$	$2^8 2^{2k}$	$6+3\cdot 7\cdot 2^{-1}x+2^{-1}x^2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{35}$	8	2	$-5+3\cdot 2^2x$	$1-x+2x^2$	$-2^{-1}x+2^{-1}x^2$	$2^8 2^{2k}$	$6+3\cdot 7\cdot 2^{-1}x+2^{-1}x^2$
S_3	$\mathcal{K}^{36}$	8	$-2+2^6x$	0	0	0	$2^8 2^k$	$6+2^6x$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{38}$	8	$-2+2^6x$	0	0	0	$2^8 2^k$	$6+2^6x$
	$\mathcal{K}^{39}$	8	$-2+2^5x$	2^3x	0	0	$2^8 2^k$	$6+5\cdot 2^3x$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{41}$	8	$-2+2^5x$	2^3x	0	0	$2^8 2^k$	$6+5\cdot 2^3x$
S_4	$\mathcal{K}^{42}$	8	$-2+2^6x$	0	0	0	$2^8 2^k$	$6+2^6x$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{44}$	8	$-2+2^6x$	0	0	0	$2^8 2^k$	$6+2^6x$
	$\mathcal{K}^{45}$	8	$-2+2^5x$	2^3x	0	0	$2^8 2^k$	$6+5\cdot 2^3x$
S_5	$\mathcal{K}^{46}$	8	$-2+2^6x$	0	0	0	$2^8 2^k$	$6+2^6x$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{49}$	8	$-2+2^6x$	0	0	0	$2^8 2^k$	$6+2^6x$
	$\mathcal{K}^{50}$	8	$-2+2^5x$	2^3x	0	0	$2^8 2^k$	$6+5\cdot 2^3x$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\mathcal{K}^{51}$	8	$-2+2^5x$	2^3x	0	0	$2^8 2^k$	$6+5\cdot 2^3x$

Table A.1: Character degrees of 2-groups of coclass 2

$[g_2, V] = \langle t_1 \rangle$ and $[g_1 g_2, V] = \langle t_1^{2^{k+3}} \rangle$. As the subgroup $V = \langle g_3 \rangle \times \langle g_4 \rangle$ is isomorphic to $C_{2^{k+5}} \times C_2$, the following homomorphisms

$$\begin{array}{rccccc} \varphi_1: & V & \to & \mathbb{C}^*, & g_3 & \mapsto & \zeta_{2^{k+5}}, \\ & & & & g_4 & \mapsto & 1 \end{array} \quad \text{and} \quad \begin{array}{rccccc} \varphi_2: & V & \to & \mathbb{C}^*, & g_3 & \mapsto & 1, \\ & & & & g_4 & \mapsto & -1 \end{array}$$

are well defined, where $\zeta_{2^{k+5}}$ denotes a primitive 2^{k+5}-th root of unity. As a consequence the group of linear characters $\operatorname{Lin} V$ is the direct product $\langle \varphi_1 \rangle \times \langle \varphi_2 \rangle$. For $\varphi \in \operatorname{Lin} V$ let c_φ, c_V, a_V and $a_{\varphi,V}$ be defined as in Chapter 7. Then Lemma 80 yields

$$\begin{aligned} &\{\varphi \in \operatorname{Lin} V \mid \varphi\uparrow_V^H \in \operatorname{Irr} H\} = \{\varphi \in \operatorname{Lin} V \mid c_\varphi = c_V\} \\ &= \{\varphi \in \operatorname{Lin} V \mid \langle t_1^{2^{k+3}} \rangle \not\leq \operatorname{Ker} \varphi\} = \langle \varphi_1, \varphi_2 \rangle \setminus \langle \varphi_1^2, \varphi_2 \rangle. \end{aligned}$$

Since V is a normal subgroup of G, the functions a_V and $a_{\varphi,V}$ are constant 1 and hence $a_V * a_{\varphi,V} = 1 \cdot 1 + \cdots + 1 \cdot 1 = 4$ for all linear characters φ of V. It follows by Theorem 83 that $\mathrm{N}_{2^2}(G) = \frac{1}{4} |\langle \varphi_1, \varphi_2 \rangle \setminus \langle \varphi_1^2, \varphi_2 \rangle| = \frac{1}{4}(2 \cdot 2^{k+5} - 2 \cdot 2^{k+4}) = 2^{k+3}$.

Appendix B

Counterexample to Theorem 23 of [9]

In this chapter we give a counterexample to

Theorem 111 ([9, Theorem 23])**.** *Let S be a uniserial p-adic pre-space group of dimension d, say, let T be a translation subgroup of S with corresponding point group $P = S/T$. Further, let $n \geq ed$ where $|P| = p^e$, and let T_n denote the $(n+1)$-th term of the lower P-central series of T. If the nilpotency class of P is sufficiently large, then for every $(\beta, \epsilon) \in \mathrm{Comp}(P, T/T_n)$ there exists $(\beta, \delta) \in \mathrm{Comp}(P, T)$ such that (β, ϵ) and $(\beta, \delta_{T/T_n})$ induce the same automorphism of $H^2(P, T/T_n)$.*

First, we construct a pro-2-group of coclass 3. We write D_8 to denote the dihedral group of order 8 given by the following pc-presentation

$$D_8 := \langle a, b \mid a^2 = 1, b^4 = 1, b^a = b^{-1} \rangle.$$

Define an action of D_8 on the $\mathbb{Z}_2$-module $\mathbb{Z}_2[i]$ by setting $1.a := 1$, $i.a := -i$ and $1.b := i$, $i.b := i \cdot i = -1$. Evidently, this action is uniserial. Then $S := D_8 \ltimes \mathbb{Z}_2[i]$ is a pro-2-group of coclass 3. Let k denote a nonnegative integer and put $T := 1 \ltimes 2^k \mathbb{Z}_2[i] \trianglelefteq S$ and $P := S/T$. Since D_8 acts uniserially on $\mathbb{Z}_2[i]$, the nilpotency class of P is at least $\log_2 |\mathbb{Z}_2[i]/2^k \mathbb{Z}_2[i]| = 2 \cdot k$. To ease the notation, identify P with its pc-presentation w.r.t. the polycyclic sequence $\mathcal{G} := ((a, 0)T, (b, 0)T, (1, 1)T, (1, i)T)$, in particular $P = \langle x_1, \ldots, x_4 \mid \Omega \rangle$ where Ω consists of the relators

$$\begin{array}{llll}
\omega_{1,1} = x_1^2, & \omega_{2,2} = x_2^4, & \omega_{3,3} = x_3^{2^k}, & \omega_{4,4} = x_4^{2^k}, \\
\omega_{1,2} = x_1^{-1} x_2 x_1 x_2, & \omega_{1,3} = x_1^{-1} x_3 x_1 x_3^{2^k - 1}, & \omega_{1,4} = x_1^{-1} x_4 x_1 x_4, & \omega_{2,3} = x_2^{-1} x_3 x_2 x_4^{2^k - 1}, \\
\omega_{2,4} = x_2^{-1} x_4 x_2 x_3, & \omega_{3,4} = x_3^{-1} x_4 x_3 x_4^{2^k - 1}. & &
\end{array}$$

Now we determine $B^2(P, T)$ and $Z^2(P, T)$. For this purpose, let F denote the free group on $\{x_1, \ldots, x_4\}$, and for $\gamma \in Z^2(P, T)$ and $s = (s_1, s_2, s_3, s_4) \in T^4$ let δ_γ, τ_s and δ_s be as in Section 4.2.2. Put $(g_1, g_2, g_3, g_4) := \mathcal{G} = ((a, 0)T, (b, 0)T, (1, 1)T, (1, i)T)$. Recall that the group homomorphism $\tau_s : F \to E(0)$ is defined by the images $\tau_s(x_i) = (g_i, s_i)$, and that δ_s is the composition of τ_s and the projection $E(0) \to T$, that is, $\tau_s(\omega) = (\omega(\mathcal{G}), \delta_s(\omega))$ for $\omega \in F$. It is straightforward to obtain the following images of the relators $\omega_{i,j}$ under τ_s for $s = (s_1, \ldots, s_4) \in T^4$:

$$\begin{array}{lll}
\delta_s(\omega_{1,1}) = s_1.g_1 + s_1, & \delta_s(\omega_{2,2}) = s_2.(1 + g_2 + g_2^2 + g_2^3), & \delta_s(\omega_{3,3}) = 2^k s_3, \\
\delta_s(\omega_{4,4}) = 2^k s_4, & \delta_s(\omega_{1,2}) = s_1.(-1 + g_2) + s_2.(1 + g_1 g_2), & \delta_s(\omega_{1,3}) = s_3.(2^k - 1 + g_1), \\
\delta_s(\omega_{1,4}) = s_4.(1 + g_3), & \delta_s(\omega_{2,3}) = -s_2 + s_3.g_2 + (2^k - 1) \cdot s_4, & \delta_s(\omega_{2,4}) = s_3 + s_4.g_2, \\
\delta_s(\omega_{3,4}) = 2^k \cdot s_4. & &
\end{array}$$

Further, denote

$$\begin{array}{ll}
\tau_2 : Z^2(P,T) \to T^{10}, & \gamma \mapsto (\delta_\gamma(\omega_{1,1}), \delta_\gamma(\omega_{2,2}), \ldots, \delta_\gamma(\omega_{2,4}), \delta_\gamma(\omega_{3,4}), \\
\sigma : T^4 \to T^{10}, & s \mapsto (\delta_s(\omega_{1,1}), \delta_s(\omega_{2,2}), \ldots, \delta_s(\omega_{2,4}), \delta_s(\omega_{3,4})),
\end{array}$$

and put $Z_2 := \operatorname{Im} \tau_2$ and $B_2 := \operatorname{Im} \sigma$. Then by Lemma 28 the torsion subgroup of T^{10}/B_2 is the factor group Z_2/B_2 which is isomorphic to $H^2(P,T)$ by Lemma 25. Recall that the action of P on $T = 1 \ltimes 2^k \mathbb{Z}_2[i]$ is induced by the action of D_8 on $\mathbb{Z}_2[i]$. Since $\mathbb{Z}_2[i] \to T$, $t \mapsto (1, 2^k \cdot t)$ is a D_8-module isomorphism, we may identify

$$T = \mathbb{Z}_2[i].$$

Let $T \geq T_1 \geq T_2 \geq \cdots$ be the lower P-central series of T, and note that $T_2 = 2 \cdot T$. Then Lemma 27 and the natural $\mathbb{Z}_2$-module isomorphism $T \to \mathbb{Z}_2^2$, $1 \mapsto (1,0)$, $i \mapsto (0,1)$ yield the following transformation matrix of the $\mathbb{Z}_2$-module homomorphism σ

$$S := \left(\begin{array}{cccccccccccccccccccc}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k - 2 & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 2^k - 1 & \cdot & \cdot & 1 & 2^k & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k - 1 & -1 & \cdot & \cdot & 2^k
\end{array}\right),$$

where $\cdot$ represent 0. Assume that k is at least 1. Then the Smith normal form of S is the diagonal matrix with diagonal entries 1, 1, 1, 1, 1, 2, 2, 2^k, in particular $H^2(P,T) \cong C_2^2 \times C_{2^k}$. Further, the row space of S is generated by the rows of the following matrix in row echelon form

$$\left(\begin{array}{cccccccccccccccccccc}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & 2^k & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & 2^k \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2^k & 2^k & \cdot & \cdot & \cdot & \cdot & 2^k - 2 & 2 & \cdot & \cdot & 2^k - 2 & \cdot & 2 & 2^k & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & 2^k - 1 & \cdot & 1 & 2^k & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k - 1 & -1 & \cdot & \cdot & 2^k \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right).$$

Hence Z_2 corresponds to the row space of

$$Z := \left(\begin{array}{cccccccccccccccccccc}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2^{k-1} & 2^{k-1} & \cdot & \cdot & \cdot & \cdot & 2^{k-1} - 1 & 1 & \cdot & \cdot & 2^{k-1} - 1 & \cdot & 1 & 2^{k-1} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & 2^k - 1 & \cdot & 1 & 2^k & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2^k - 1 & -1 & \cdot & \cdot & 2^k \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right).$$

Let e denote the 2-logarithm of $|P|$, that is, $e = 3 + 2k$. In what follows, we determine the P-endomorphisms $\mathrm{End}_P(T/T_{2e})$ and we show that $\mathrm{End}_P(T/T_{2e})$ does not stabilize $Z^2(P, T/T_{2e})$. Denote $t_1 := 1 + T_{2e}$ and $t_2 := i + T_{2e}$. Since the action of P on $T = \mathbb{Z}_2[i]$ is induced by the action of D_8, we have $\mathrm{End}_P(T/T_{2e}) = \mathrm{End}_{D_8}(T/T_{2e})$. Further, T is generated by 1 and T_1, and the stabilizer of 1 in D_8 is $\langle a \rangle$. As the $\mathbb{Z}_p$-submodule $C_{T/T_{2e}}(\langle a \rangle) \leq T/T_{2e}$ of fixed points under the action of $\langle a \rangle$ equals $\langle t_1, 2^{e-1}t_1 + 2^{e-1}t_2 \rangle$, it follows by Lemma 48 that the P-endomorphisms $\mathrm{End}_P(T/T_{2e}) = \mathrm{End}_{D_8}(T/T_{2e})$ are generated by the identity map and the endomorphism φ which is defined by $\varphi(t_1) = 2^{e-1}t_1 + 2^{e-1}t_2$ and $\varphi(t_2) = 2^{e-1}t_1 + 2^{e-1}t_2$.
Now let π be the group homomorphism $Z^2(P, T) \to T$, $\gamma \mapsto \delta_\gamma(\omega_{1,4})$, and note that the image of π can be read off from the 13-th and 14-th column of the matrix Z. This yields $\mathrm{Im}\,\pi = \langle t_1 \rangle$ and hence the factor module $\mathrm{Im}\,\pi + T_{2e}/T_{2e}$ is not invariant under the action of $\varphi \in \mathrm{End}_P(T/T_{2e})$. Thus $\mathrm{End}_P(T/T_{2e})$ does not stabilize $(Z^2(P,T))_{T/T_{2e}}$. Recall that $\mathrm{End}_P(T/T_{2e}) \to \mathrm{Comp}(P, T/T_{2e})$, $\epsilon \mapsto (1, \epsilon)$ is a group homomorphism and that $(1, \epsilon)$ acts on $Z^2(P, T/T_{2e})$ by ϵ. It follows that the $\mathrm{Comp}(P,T)$-module $(Z^2(P,T))_{T/T_{2e}}$ is not stabilized by $\mathrm{Comp}(P, T/T_{2e})$. This contradicts Theorem 111.

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Nomenclature

$\alpha \otimes \mathrm{id}$	$\alpha \otimes \mathrm{id} : C^1(P,T) \otimes \mathbb{Q}_p \to B^2(P,T) \otimes \mathbb{Q}_p$ . . . 72
$\operatorname{Aut} \gamma$	automorphism groups of a 2-cocycle γ . . . 17
$\operatorname{Aut} G$	automorphism group of a group G . . . 15
$\mathcal{B}_{m,i}(S)$	i-th branch of the shaved coclass tree $\mathcal{T}_m(S)$ . . . 36
$\mathcal{B}_i(S)$	i-th branch of the coclass tree $\mathcal{T}(S)$ . . . 36
$\chi \uparrow_U^G$	character induced by an character χ of $U \leq G$ . . . 54
$\mathrm{Comp}(P,T)$	group of compatible pairs of P and T . . . 15
$\mathrm{DCM}(U_j, U_i)$	$= \{a : U_j \backslash G / U_i \to \{0,1\}\}$, double coset maps . . . 55
$\delta'_{2,k}$	$\delta'_{2,k} : H^3(P, T_n) \to H^2(P, T/T_{n+kd})$, right inverse of the connecting homomorphism . . . 31
δ_γ	$\delta_\gamma : F(x_1, \ldots, x_m) \to T$, where $\gamma \in Z^2(\langle g_1, \ldots, g_m \rangle, T)$ . . . 21
$\mathrm{dep}_{\mathcal{G}}(g)$	depth of g with respect to the polycyclic series $\mathcal{G}$ . . . 20
$\operatorname{End}_P^\beta A_k$	$= \mathrm{Hom}_P(A_k, A_k^{(\beta)})$ . . . 34
$\operatorname{End}_P^\beta T$	$= \mathrm{Hom}_P(T, T^{(\beta)})$ . . . 34
$\exp G$	exponent of a group G . . . 12
$\exp_{\mathcal{G}}(g)$	exponent vector of g with respect to the polycyclic sequence $\mathcal{G}$ . . . 20
Γ	$= \mathrm{Comp}(P,T)$ . . . 33
$\gamma_i(G)$	i-th term of the lower central series of the group G . . . 27
Γ_k	$\mathrm{Comp}(P, A_k)$ . . . 33
ι_k	$\iota_k : p^k T_{n-1} \to \operatorname{Ker} \phi_k,\ t \mapsto (1,1,\delta_t)$ . . . 73
λ_k	$\lambda_k : \Gamma_0 / \operatorname{Im}(\pi_0 \circ \varrho) \to \Gamma_k / \operatorname{Im}(\pi_k \circ \varrho)$ . . . 35
$\langle\langle U \rangle\rangle$	normal closure of a subgroup U in its overgroup . . . 23

$\mathrm{lead}_{\mathcal{G}}(g)$	leading exponent of g with respect to the polycyclic series $\mathcal{G}$ ... 20
$\mathrm{Lin}\, G$	class of linear characters of a group G ... 53
$\mathcal{E}_c(v,w)$	strict generators sequence ... 44
$\mathcal{F}(\gamma,\delta,n)$	coclass family defined by $\gamma \in Z^2(P,T)$, $\delta \in C^2(P,T)$ and $n \in \mathbb{N}$ ... 30
$\mathcal{G}(p,r)$	coclass graph of p-groups of coclass r ... 28
$\mathcal{G}_m(p,r)$	shaved coclass graph ... 36
$\mathcal{T}(S)$	coclass tree induced by S ... 36
$\mathcal{T}_m(S)$	shaved coclass tree induced by S ... 36
$\mathrm{cl}(G)$	nilpotency class of the finite p-group G ... 27
$\mathrm{Stab}_G(m)$	stabilizer subgroup of m in G ... 16
μ_k	$= \mu(n)_k : H^2(P,T/T_n) \to H^2(P,T/T_{n+kd})$ ... 31
$\mathrm{N}_l(G)$	number of irreducible characters of G of degree p^l ... 53
$\mathrm{norm}(g)$	normal form of an element g with respect to $\mathcal{G}$ ... 23
ω	$\omega' : \mathrm{End}_P(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\},\ \varphi \otimes k \mapsto \sup\{m \mid \mathrm{Im}\,\varphi \subseteq T_m\} + \nu_p(k) \cdot d$ ... 70
$\Omega(\mathcal{G})$	set of relators of $\mathrm{pc}(\mathcal{G})$ ... 20
ω_i	$\omega_i : C^i(P,T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\},\ \psi \otimes k \mapsto \sup\{m \mid \mathrm{Im}\,\psi \subseteq T_m\} + \nu_p(k) \cdot d$ ... 70
$\mathrm{pc}\langle x_1,\dots,x_m \mid R\rangle$	shorten notation of pc-presentation ... 20
$\mathrm{pc}(\mathcal{G})$	pc-presentation of $G = \langle \mathcal{G} \rangle$ with respect to $\mathcal{G}$ ... 20
ϕ_k	$\phi_k : \mathrm{Aut}\,\gamma \to \mathrm{Aut}\,\gamma_{A_k},\ (\beta,\epsilon,\psi) \mapsto (\beta,\epsilon_{A_k},\psi_{A_k})$ ... 69
$\phi_{c,k}$	$\phi_{c,k} : \mathrm{Aut}\,\delta_k \to \mathrm{Aut}\,\gamma_{A_{k-c}},\ (\beta,\epsilon,\psi) \mapsto (\beta,\epsilon_{A_{k-c}},\psi_{A_{k-c}})$ ... 69
π_k	$\pi_k : \Gamma \to \Gamma_k,\ (\beta,\epsilon) \mapsto (\beta,\epsilon_{A_k})$ ... 34
ψ_γ	$\psi_\gamma \in C^1(P,T)$ of maximal value $\omega_1(\psi_\gamma \otimes 1)$ with $\alpha(\psi_\gamma) = o_\gamma \cdot \gamma$ ... 72
ϱ	$\varrho : 1 + \max\{a,b\} \cdot \mathrm{End}_P\, T \to \Gamma,\ \epsilon \mapsto (1,\epsilon)$ ... 34
ϱ_k	$\varrho_k : E_k \to \Gamma_k,\ \epsilon \mapsto (1, 1+\epsilon)$ ... 34
ϑ_k	$\vartheta_k : 1 + p^k f\, \mathrm{End}_P\, T \to \mathrm{Ker}\,\phi_k,\ 1+\varphi \mapsto (1, 1+\varphi, o_\gamma^{-1}\psi_\gamma.\varphi)$ ... 73
a	$= a(n) = \max\{\exp H^2(P,T), \exp H^3(P,T_n)\}$ ... 31
A_k	T/T_{n+kd} ... 32

$A_k^{(\beta)}$	$= T^{(\beta)}/T_k^{(\beta)}$ ... 34
a_U	$a_U : U\backslash G/U \to \mathbb{N},\ UgU \mapsto [U^g : (U \cap U^g)(U^g)']$ ... 54
$a_{\chi,W}$	$a_{\chi,W} : W\backslash G/U \to \{0,1\}$ ... 54
b	$= b(n) = \exp H^1(P_{t_0}, T_n)$ ... 31
$B^n(G,A)$	group of n-coboundaries ... 9
B_k	$= \operatorname{Ker} \phi_k$ ... 74
c	$c \in \mathbb{N}$ with $p^{c-1} = \max\{a,b\}^2$ ... 69
$C^n(G,A)$	group of n-cochain maps ... 9
c_U	$c_U : U\backslash G/U \to \{0,1\}$ characteristic function whose support is $\{U1U\}$ ... 53
$C_V(H)$	elements of the H-set V fixed by H ... 33
c_χ	$c_\chi : U\backslash G/U \to \{0,1\}$, where $\chi \in \operatorname{Lin} U$ and $U \le G$ ... 53
$E(\gamma)$	group extension defined by the 2-cocycle γ ... 15
E_k	$= (p^k E)_{A_k}$ ... 34
$F(x_1, \ldots, x_m)$	free group on the set $\{x_1, \ldots, x_m\}$ ... 21
G_m	stabilizer subgroup of m in G ... 31
$H^n(G,A)$	n-th cohomology group ... 9
n'	$n' := n/d$ ... 43
o_γ	order of $\gamma + B^2(P,T)$ ... 72
T_+	$\mathbb{Z}_p$-module written additively ... 43
$T_k^{(\beta)}$	$= T_k$ as set, P-module structure induced by $\beta \in \operatorname{Aut} P$ ... 34
X_k	polycyclic sequence of G_k ... 43
X_{A_k}	induced polycyclic sequence of $A_k \le G_k$ ... 43
$Z^n(G,A)$	group of n-cocycles ... 9

Index

Curriculum Vitae

Personal Data

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Education

08.1996-06.2005	Gesamtschule Hardt, Mönchengladbach
June 24, 2005	Abitur (grade 1.0)
04.2006-05.2010	RWTH Aachen: Student of mathematics with economics as minor subject
May 11, 2010	"Diplom" in mathematics

Awards

June 6, 2009	*Schöneborn Preis* from proRWTH for outstanding achievements during undergraduate study ("Vordiplom"), endowed with prize money of 500 Euros
June 17, 2011	*Springorum-Denkmünze* from proRWTH for outstanding achievements during graduate study ("Diplom")

Positions

Since 07.2010	Teaching and Research assistant at TU Braunschweig

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07.2005-03.2006	Compulsory civilian service at Ambulante Dienste, Mönchengladbach

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